3-D Mathematical Preliminaries

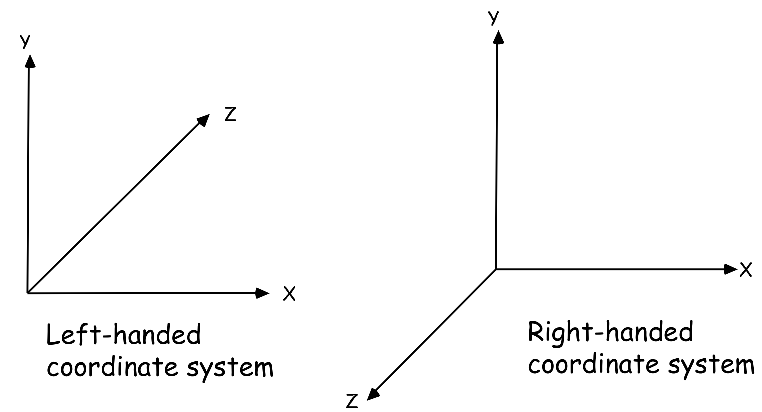
First, let's finish some 2D stuff

- Questions?

2D Rotations

- "Square" vs "Arbitrary object"
  - Pick rotation and reference points

3D Coordinate Systems



3-D Vectors

Have length and direction

$V = [x_v, y_v, z_v]$

Length is given by the Euclidean Norm

$||V|| = \sqrt{(x_v^2 + y_v^2 + z_v^2)}$

Dot Product $V \cdot U = [x_v, y_v, z_v] \cdot [x_u, y_u, z_u]$

$= x_v x_u + y_v y_u + z_v z_u$

$= ||V|| \; ||U|| \cos ß$

Cross Product $V \times U = [v_y u_z - v_z u_y, \; -v_x u_z + v_z u_x, \; v_x u_y - v_y u_x]$

$V \times U = - (U \times V)$

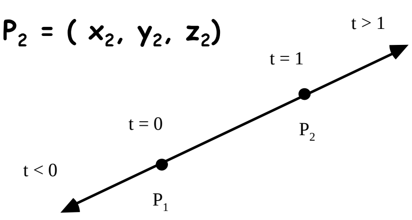
Parametric Definition of a Line

Given two points: $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$

$x = x_1 + t (x_2 - x_1)$

$y = y_1 + t (y_2 - y_1)$

$z = z_1 + t (z_2 - z_1)$



Given a point P_1 and a vector $V = [x_v, y_v, z_v]$

$x = x_1 + t x_v$, $y = y_1 + t y_v$, $z = z_1 + t z_v$

Short form: $L = P_1 + t[P_2 - P_1]$ or $L = P_1 + Vt$

Equation of a plane:

$Ax + By + Cz + D = 0$

Normalized Form: $A'x + B'y + C'z + D' = 0$

where $A' = A/d$, $B' = B/d$, $C' = C/d$, $D' = D/d$

$d = \sqrt{(A^2 + B^2 + C^2)}$

Distance between a point and the plane is given by

$A'x + B'y + C'z + D'$ (sign indicates which side)

[A, B, C] is the normal vector

Proof: Given P_1 and P_2 in the plane, $[P_2 - P_1]$ is in the plane and

$$[A,B,C] \cdot [P_2 - P_1] = (Ax_2 + By_2 + Cz_2) - (Ax_1 + By_1 + Cz_1) = (-D) - (-D) = 0$$

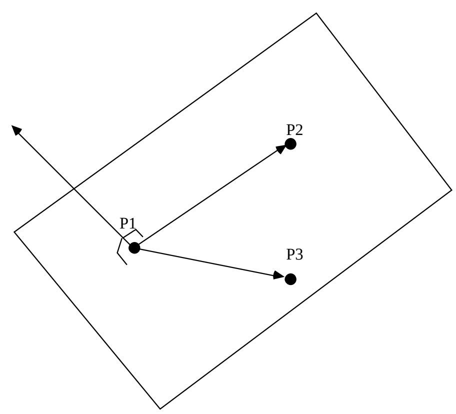
Derivation of Plane Equation

To derive equation of the plane given three points: **P_1, P_2, P_3**

$[P_3 - P_1] \times [P_2 - P_1] = N$, orthogonal vector

Given a point **$P = (x,y,z)$**

$N \cdot [P - P_1] = 0$ if **P** is in the plane.



Basic Transformations

- **Translation**
- **Scale**
- **Rotation**
- **Shear**

Translation

TP = (x + t_x, y + t_y, z + t_z)

$$\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

T **P**

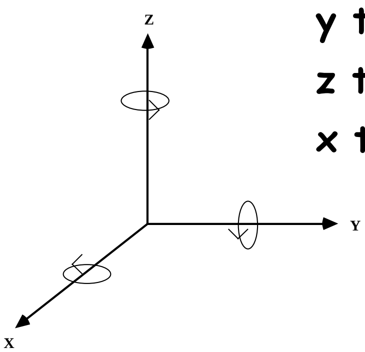
Scale

$SP = (s_x x,\ \ s_y y,\ \ s_z z)$

$$
\begin{matrix}
(s_x & 0 & 0 & 0) & (x) \\
(0 & s_y & 0 & 0) & (y) \\
(0 & 0 & s_z & 0) & (z) \\
(0 & 0 & 0 & 1) & (1)
\end{matrix}
$$

Rotations

Axis of rotation is	Direction of positive rotation is
x	y to z
y	z to x
z	x to y



Rotations

About the z axis

$$R_z(\beta)\,P = \begin{pmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

About the x axis

$$R_x(\beta)\,P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta & 0 \\ 0 & \sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

About the y axis

$$R_y(\beta)\,P = \begin{pmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Shears

$$SH_{xy}P = \begin{pmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Homogeneous 3-D Coordinates

(**T** is any transformation, **P** is any point)

TP = T(x,y,z,1) = (x', y', z',w)

Homogenize the result:

P_h = (x'/w, y'/w, z'/w,1)