

## Homework 5 – extended due date Tuesday, March 13

### Part I

**Problem 1** Prove that, for all  $n > 1$ ,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \frac{13}{14}$$

Hint: prove a stronger inequality.

**Problem 2** Prove that the regions formed by  $n$  circles in the plane can be colored with two colors such that regions sharing an edge are colored differently.

**Problem 3** Consider  $n \geq 3$  lines in general position in the plane. Prove that at least one of the regions formed is a triangle.

### Part II

**Problem 4** The Fibonacci numbers are defined recursively by  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for every  $n \geq 2$ . Prove that, for every  $n \geq 3$ ,

$$F_{2n+1} = (F_n)^2 + (F_{n+1})^2$$

Hint: prove at the same time that  $F_{2n} = (F_{n+1})^2 - (F_{n-1})^2$ .

**Problem 5** Define a function  $f$  of two non-negative integer variables by

$$f(n, k) = \begin{cases} 1, & \text{if } n = 0 \\ 1, & \text{if } k = 0 \\ f(n-1, k) + f(n, k-1), & n > 0 \text{ and } k > 0 \end{cases}$$

Prove that

$$f(n, k) = \frac{(n+k)!}{n!k!}$$

(remember that  $n! = 1 \cdot 2 \cdot \dots \cdot n$  for  $N \geq 1$  and  $0! = 1$ ). Hint: use induction on  $k$  inside an induction on  $n$ .

**Problem 6** Prove that

$$(1 + a_1) \cdot (1 + a_2) \cdots (1 + a_n) \geq 2^n$$

for every positive real numbers  $a_1, a_2, \dots, a_n$  such that  $a_1 \cdot a_2 \cdots a_n = 1$ .

Hint: use the inequality between geometric and arithmetic means.

### Extra Credit

**Problem 7** Prove Problem 6 without using the inequality between geometric and arithmetic means.

**Problem 8** For every  $n \geq 1$ , let  $A_n = \{1, 2, \dots, 2n-1, 2n\}$ . Prove that every subset of  $n+1$  numbers from  $A_n$  contains one which divides another.