CS 1050 Section B
Sample Final Exam

5 points: Prove that if x is a rational number, then $x\sqrt{2}$ is an irrational number. You may use the fact that $\sqrt{2}$ is irrational.
5 points: Prove that $1 + 2 + 3 + \cdots + n$ is divisible by n if and only if n is odd.
10 points: What is $90^{1584} \pmod{11}$? (Hint: Use Fermat's Little Theorem)
10 points: Show that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(a, b) = 2^a 3^b$ is one-to-one.
10 points: Prove that $(1 + x)^n > 1 + nx$ for every real $x > 0$ and integer $n > 1$.
15 points: Let $F_o, F_1, F_2, \ldots$ denote the Fibonacci numbers, defined by $F_o = 0$, $F_1 = 1$, and, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Use the second principle of mathematical induction to prove that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right].$$

10 points: The numbers 1447, 1606, and 1231 have something in common: each is a 4-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?
5 points: How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are there?
5 points: How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ have an even number of elements? Explain why.
5 points: Let $x_1, x_2, \ldots, x_{m+1}$ be any $m + 1$ integers. Prove that there must exist a pair whose difference is divisible by m.
10 points: Let $y_1, y_2, \ldots, y_m$ be any m integers. Prove that there must exist a sequence $y_k, y_{k+1}, \ldots, y_{k+l}$ of consecutively indexed y_i's that sum up to a multiple of m.
5 points: Suppose we roll three unbiased dice, each with faces numbered 1 through 6. What is the probability that all dice show the same number?
5 points: Suppose that A and B are events in a probability space, and that $\Pr[A] = 0.5$, $\Pr[B] = 0.2$, and $\Pr[A \cup B] = 0.6$. What is $\Pr[A \cap B]$?
10 points: Let p and q be prime numbers, and $n = pq$. What is the probability that an integer chosen randomly from the set $\{1, \ldots, n\}$ is *not* divisible by either p or q?
5 points: Consider a graph G with 11 nodes and each node of degree 6. How many edges does graph G have?
10 points: Prove that if a tree has a vertex of degree $d \geq 2$, then it must have at least d leaves (vertices of degree one).