1: Since each A_i is countable, the elements of A_i can be ordered as $\{a_{i1}, a_{i2}, \ldots\}$. Hence $A_1 \bigcup A_2 \bigcup \ldots = \{a_{11}, a_{12}, \ldots, a_{21}, a_{22}, \ldots, a_{31}, a_{32}, \ldots\}$ (there might be some repititions in the list, but that only decreases the number of elements in the set). This set can be shown to be countable just as we show $N \times N$ is countable.

2: We know that Z is countable. Also, the product of two countable sets is countable (similar to proving $N \times N$ is countable.) There is a simple bijection from S to $Z \times Z$: Let $f(a + ib) = (a, b)$. Then f is the required bijection. Therefore, S is countable.

3: Define the dice as follows

- Dice A: 4, 5, 10
- Dice B: 3, 7, 9
- Dice C: 2, 6, 12
- Dice D: 1, 8, 11

All 3×3 outcomes when you toss a pair of dice are equally probable. $A > B$ since in $5 (> 9/2)$ of the possible outcomes (4, 3), (5, 3), (10, 3), (10, 7), (10, 9), A beats B. (Note: This is not the only possible solution)

4:

- Dice A: $Pr[3] = 1$
- Dice B: $Pr[2] = p = (\sqrt{5} - 1)/2$, $Pr[5] = 1 - p$
- Dice C: $Pr[1] = 1 - p$, $Pr[4] = p$

(Note: This is not the only possible solution)

5: Let $q = 1 - p$ (for simplicity of notation).
Probability that a step is HT or $TH = 2pq$.
Probability that a step is HH or $TT = p^2 + q^2$.
Probability that we stop after the i^{th} step $p_i = (p^2 + q^2)^{i-1}(2pq)$.
Therefore, $Pr[E] = \bigcup_{i=2,4,\ldots} p_i$.
Also, E and H_f are independent events since $Pr[E \bigcap H_f] = Pr[E]Pr[H_f]$.
Therefore $Pr[E|H_f] = Pr[E]$, and $Pr[H_f|E] = Pr[H_f] = 0.5$.