Linear interpolation

'Real' functions are rarely given by an explicit formula allowing to evaluate them anywhere. More frequently, only 'sample' values of the function are given at certain points and, in order to estimate the value at some place which is not a sample, we need to somehow combine the available information. This is the goal of interpolation. By doing interpolation one can, for example, build a complete elevation map of a terrain when only an array of height values is given (this is what happens in practice: height cannot be measured everywhere; no matter how you do it, you end up with only finite number of measurements). We used interpolation to estimate the color over a triangle when colors at vertices are given, or normals/depths over the triangle from normals/depths at the vertices.

Linear interpolation from vertices of a triangle

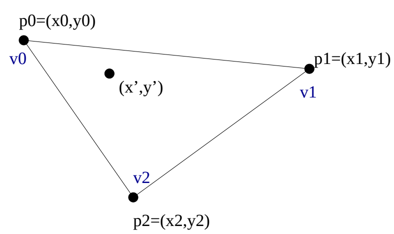

Figure 1: Linear interpolation from triangle's vertices: notation.

Here is how the variant of linear interpolation we used works. Let's say that we have a triangle as shown in Figure 1. It has vertices $p_0 = (x_0, y_0)$, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$. At each vertex we have a value, let's say that they are v_0, v_1 and v_2. There is exactly one linear function which takes the value of v_0 at p_0, v_1 at p_1 and v_2 at p_2 (We can prove that by 'embedding' our triangle and the values at vertices in 3D: say that the triangle lies in the groung plane; lift p_0 to height v_0, p_1 to height v_1 and p_2 to height v_2. This yields 3 points in 3-space. Now, find the plane passing through the three points. This plane is the graph of the linear function we are looking for). Here is how to compute it algebraically.

We seek a function of the form $f(x, y) = Ax + By + C$ $(A, B, C$ to be determined). The requirement that the values at the vertices are v_0, v_1 and v_2 leads to three linear equations

$$Ax_0 + By_0 + C = v_0$$
$$Ax_1 + By_1 + C = v_1$$
$$Ax_2 + By_2 + C = v_2.$$

We can solve them, obtaining A, B and C. So, for example, to compute the interpolated value at the point $p' = (x', y')$ we need to evaluate $f(p') = f(x', y') = A * x' + B * y' + C$.

Alternative approach

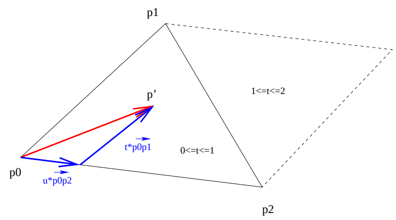

Figure 2: The second method.

One can also think of linear interpolation in the following way. We want to move from p_0 to p' by first moving some amount along the vector from p_0 to p_1 and then by some fraction of the vector from p_0 to p_2 (see Figure 2). This can be algebraically expressed in the following way:

$$p' = p_0 + t * p_0\vec{p}_1 + u * p_0\vec{p}_2. \tag{1}$$

Notice that since we know that p' is inside the triangle, t and u have to be nonnegative and need to satisfy $t + u \leq 1$. Basically, if either t or u were negative, we would be moving in the wrong direction (either to the left of the edge joining p_0 and p_1 or below the edge from p_0 to p_2). If $u + t > 1$, we would be moving too far (e.g. if $1 < t + u \leq 2$ then we would end up in the triangle complementing $\Delta p_0 p_1 p_2$ to a parallelogram - see the Figure). Now, since we want the interpolated function to be linear over the triangle, its increase along $t * p_0\vec{p}_1$ has to be t times the increase over $p_0\vec{p}_1$. A similar statement holds for $p_0\vec{p}_2$. Therefore, the interpolated value v' at p' can be computed as

$$v' = v_0 + t * (v_1 - v_0) + u * (v_2 - v_0).$$

Of course, we still need to compute t and u. This is easy though: from 1 we have the following equations that can be solved for t and u:

$$x' = x_0 + (x_1 - x_0)t + (x_2 - x_0)u$$
$$y' = y_0 + (y_1 - y_0)t + (y_2 - y_0)u$$