CS 4803A/8803A: Pattern Recognition

Date: 2/4/03 **Estimating Gaussian Variance**

Given n observations $\hat{y}_i$ of a Gaussian random variable y, find the maximum likelihood estimate of the variance σ of the distribution, assuming the mean μ is unknown.

$$\text{Let} \qquad L(\hat{y}_1, \hat{y}_2 \ldots \hat{y}_n; \boldsymbol{\Theta}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}\frac{(\hat{y}_i-\mu)^2}{\sigma^2}} \tag{1}$$

$$\ln(L) = -n \ln\sigma - n\ln\sqrt{2\pi} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\hat{y}_i-\mu)^2 \tag{2}$$

$$\text{Maximum likely } \widetilde{\mu} \;=\; \frac{1}{n}\sum_{i=1}^{n}\hat{y}_i \qquad \text{(Proven earlier)} \tag{3}$$

$$\text{Maximize } L: \qquad \frac{\partial}{\partial\sigma}(\ln(L)) \;=\; 0 \qquad \Rightarrow \tag{4}$$

$$0 \;=\; \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(\hat{y}_i-\mu)^2$$

$$\widetilde{\sigma}^2 \;=\; \frac{1}{n}\sum_{i=1}^{n}(\hat{y}_i-\widetilde{\mu})^2 \tag{5}$$

What is the expected value of the estimate of variance? $E[\widetilde{\sigma}^2] = \ldots$

$$\text{Let} \qquad z_k \;=\; y_k - \mu \qquad z_k \sim N(0,\sigma^2) \tag{6}$$

$$\bar{z} \;=\; \frac{1}{n}\sum_{i=1}^{n} z_i \;=\; \bar{y} - \mu \tag{7}$$

$$(y_k - \bar{y}) \;=\; (y_k-\mu) - (\bar{y}-\mu) \;=\; z_k - \bar{z} \tag{8}$$

$$E[(z_k-\bar{z})^2] = E[z_k^2] + E[\bar{z}^2] - 2E[z_k\bar{z}] \tag{9}$$

$$E[z_k^2] \;=\; \sigma^2 \tag{10}$$

$$E[\bar{z}^2] \;=\; \mathrm{var}(\bar{z}) = \frac{1}{n^2}(n\,\mathrm{var}(z_k)) = \frac{\sigma^2}{n} \tag{11}$$

$$E[z_k\bar{z}] \;=\; E[z_k\frac{z_1+\ldots+z_k+\ldots+z_n}{n}] = \frac{1}{n}E[z_k^2] \;=\; \frac{\sigma^2}{n} \qquad \text{(Why?)} \tag{12}$$

$$\Rightarrow \qquad E[(y_k-\bar{y})^2] = E[(z_k-\bar{z})^2] = \sigma^2 + \frac{\sigma^2}{n} - \frac{2\sigma^2}{n} \;=\; \frac{n-1}{n}\sigma^2 \tag{13}$$

$$E[\widetilde{\sigma}^2] = \frac{n-1}{n}\sigma^2 \tag{14}$$

Hence the estimate is biased but “most likely.” *Prepared by Aaron Bobick*