CS 4803A/8803A: Pattern Recognition

Problem Set 1
Date: January 14, 2003 **Due: start of class January 23, 2003**

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1. For each case below, are x and y statistically independent? Why or why not?

   (a) The point (x, y) is uniformly distributed in the unit square.

   (b) The point (x, y) is uniformly distributed in the unit circle (a subset of the unit square).

   (c) Let a and b be independent and both uniform on $[0, 1]$. Let $x = a + b$ and $y = a - b$.

   (d) Let a and b be independent and both Gaussian with zero mean and unit variance. Let $x = a + b$ and $y = a - b$.

   (e) Let a and b be independent and both Gaussian with zero mean, but the variance of a is 2 while the variance of b is 1. Again, let $x = a + b$ and $y = a - b$.

2. Let x and y be random variables with density functions

$$p_{x|y}(x|y) = \begin{cases} e^{-(x-y)} & y \leq x < \infty \\ 0 & x < y \end{cases}$$

   and

$$p_y(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) What is $p_{xy}(x, y)$? Specify the region where the joint density is nonzero.

   (b) Using Matlab, generate a plot of $p_{xy}(x, y)$.

   (c) Find $p_x(x)$ and plot that as well. Don't forget to specify regions of definition.

3. The components x_1 and x_2 of a two-dimensional random vector $\mathbf{x}$ are statistically independent and have the following marginal densities:

$$p_{x_1}(x_1) = \begin{cases} 1 & 0 \leq x_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   and

$$p_{x_2}(x_2) = \begin{cases} 2x_2 & 0 \leq x_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Let A be the event $x_1 \leq x_2$. What are the following?

   (a) $p_{\mathbf{x}}(\mathbf{x})$ (Plot it in 3D in Matlab.)

Write a Matlab **simulation** and verify your answer for the following:

(b) $Pr[A]$

(c) $p_{\mathbf{x}|A}(\mathbf{x}|A)$

(d) $p_{x_2|A}(x_2|A)$

4. The random variable x has density

$$p_x(x) = \begin{cases} 0.5/x & 1/e \leq x \leq e \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the density of $z = \ln(x)$?

   (b) What is the density of $y = 1/x$? What is the mode of $p_y(y)$, compared to the mode of $p_x(x)$?

   (c) For discrete random variables, the mode of the probability distribution can be interpreted as the most likely value. Does this interpretation apply equally well to continuous random variables?

5. A student takes a test to indicate whether or not the student is a nerd. The test is positive. Suppose the student population is known to contain 5% nerds (regardless of any test results). Suppose also that the nerd test is 75% accurate, with a false alarm rate of 20%. (In other words, if someone is a nerd, then the chance the test will come out positive is 75%; if they are not a nerd the chance of a positive result is 20%.) How probable is it that the student really is a nerd, given that the test is positive?

6. Below is a variation on the classic "Monty Hall" problem. To get credit for this problem, you must pose it to at least one other person who have not understood it before, and help them find the correct answer. Have them sign your problem set with a statement that indicates you have succeeded in this goal.

   Problem:

   Suppose I hide the solution to the current problem set in one of three identical boxes while you aren't looking. Then I ask you to guess which box it's in. I know which one it's in, and after you guess, I deliberately open the lid of an empty box, which is one of the boxes you have *not* chosen. Thus, the solutions are either in the box you guessed, or in the box (you didn't pick) that I didn't open.

   I then offer you the chance to change your mind: you can either keep the box you originally guessed, or choose the other unopened box. To maximize your chances of getting the problem set solution, what choice should you make? You must explain your answer.