

# Uniform B-Spline Curves

Uniform B-Spline curves are piecewise polynomial curves associated with a sequence of control points of any length. Unlike the Bezier curves, their degree is independent of the number of control points.

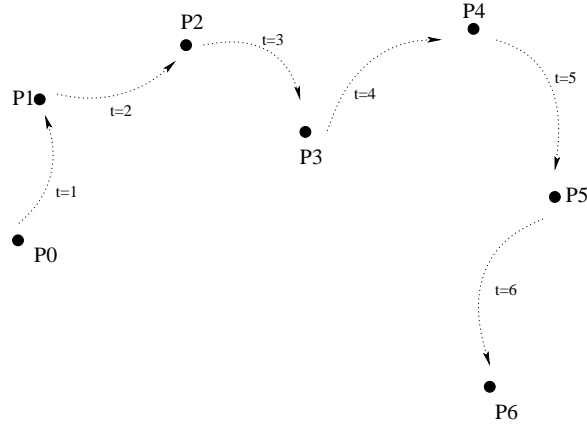


Figure 1: The ‘curve’  $B_0$ .

B-Spline curves can be constructed by recursive averaging. Start from a curve which stays one second at the first control point,  $P_0$ , then jumps to the second one,  $P_1$ , then after one second to the third etc (Figure 1). This is going to be our zero-degree B-spline curve. (we’ll generously call it a curve even though it doesn’t really look like a curve). The formula for that curve (call it  $B_0$ ) is

$$B_0(t) = P_i \text{ for } t \in [i, i + 1].$$

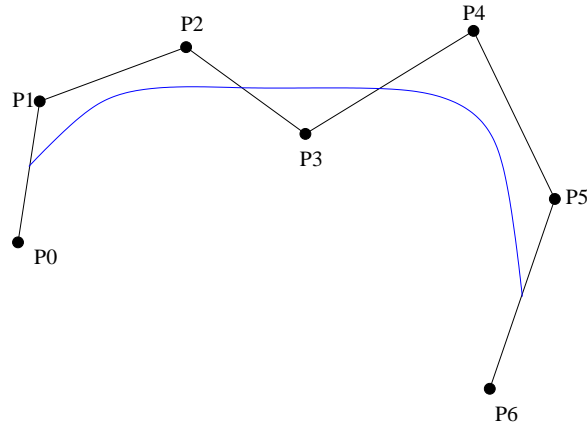


Figure 2:  $B_0$  (black) and  $B_1$  (blue).

Now we will start making the curve more regular by averaging it. We define a new curve (degree-one B-spline)  $B_1$  by setting  $B_1(t)$  to the average position of a point moving along the curve  $B_0$  over time interval  $[t, t + 1]$ . Thus,

$$B_1(t) = \int_t^{t+1} B_0(s) ds.$$

Now, it's not hard to see that  $B_1$  is a polygonal curve joining the control points (Figure 2). It's not differentiable, but it is definitely nicer-looking than  $B_0$ . The averaging procedure can be applied to  $B_1$  to lead to  $B_2$  (degree-two B-spline) and so on. Thus, we have a recursive formula

$$B_{i+1}(t) = \int_t^{t+1} B_i(s) ds.$$

## 1 Smoothness

By smoothness we'll mean the number of derivatives that can be computed for a curve. The more derivatives the smoother the curve. We'll say that a curve  $B$  is  $C^k$  if the derivatives  $B'(t), B''(t) \dots B^{(k)}(t)$  exist for all  $t$  in the parameter range, i.e. everywhere where  $B$  is defined, and they are all continuous. We'll say a curve is  $C^0$  if it's continuous.

We can now say that  $B_0$ , the zero-degree B-spline is not even  $C^0$ . The next curve in the hierarchy,  $B_1$ , is continuous but not differentiable (it has sharp corners at the control points, certainly no derivative there). Hence it's  $C^0$  but not  $C^1$ . It turns out that  $B_2$  is a  $C^1$  curve. This is a consequence of the Fundamental Theorem of Calculus, which allows to compute

$$\frac{d}{dt} B_2(t) = B_1(t+1) - B_1(t),$$

which means that the derivative can be expressed using  $B_1$  and hence is continuous since  $B_1$  is. By applying this argument once again, we can show that the degree-3 B-splines (also called cubic B-splines) are  $C^2$ , i.e. have continuous second derivative. In general,  $B_k$  is  $C^{k-1}$ .

In applications, the cubic B-spline curves are the most common: they seem to hit the right spot in the tradeoff between quality (smoothness) and simplicity (low degree). They are also well-suited for modeling based on physics. Imagine traveling in a spaceship moving along the curve  $B_k$ . For  $B_0$ , it's even hard to imagine how it's possible without seriously deforming the universe. For  $B_1$  this is also problematic, since it would require infinite acceleration at the control points; this would make the trip bumpy and unpleasant.  $B_2$  seems more plausible, at least not exposing us to infinite accelerations. However, it would not be fun at the points where the acceleration changes abruptly (imagine that you have nothing to hold to and at a certain moment the acceleration (generating a force on you because of the inertia of your body) changes very abruptly: you probably won't have time to adjust yourself and would fall down). Eventually,

$B_3$  seems to be great, offering continuous change of the acceleration and thus a pleasant journey. This is also somehow related to how we tend to drive into a sharp turn: turning the wheel too fast can be felt immediately and even make the car unstable. Thus, we try to change the turning angle (closely related to curvature and second derivative) over a longer time, making it as slowly-varying as possible. By the way, the above considerations are also important in the design of train track or highways.