# *Design and analysis of algorithms Lecture 12 & 13*

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# *DFS in directed graphs*

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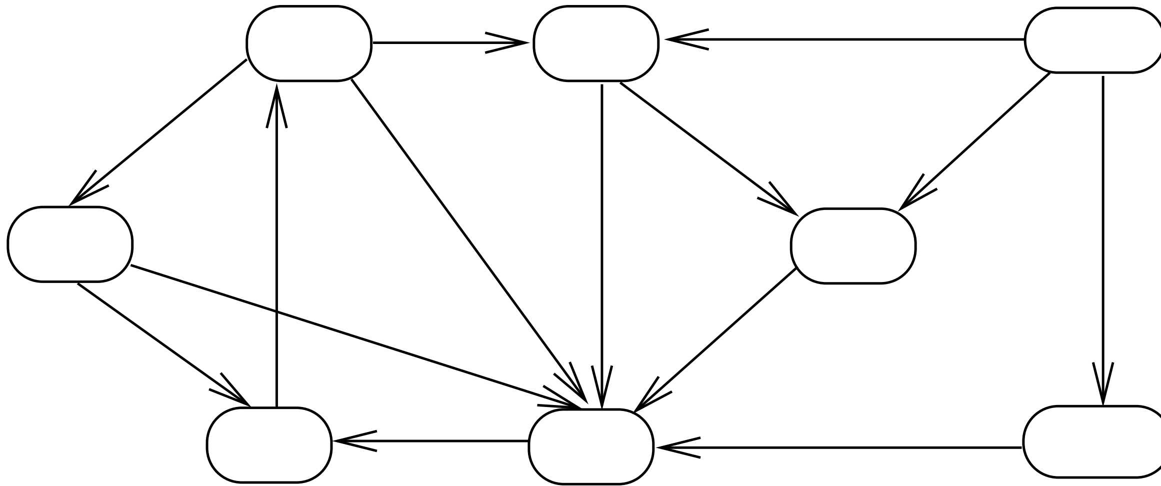

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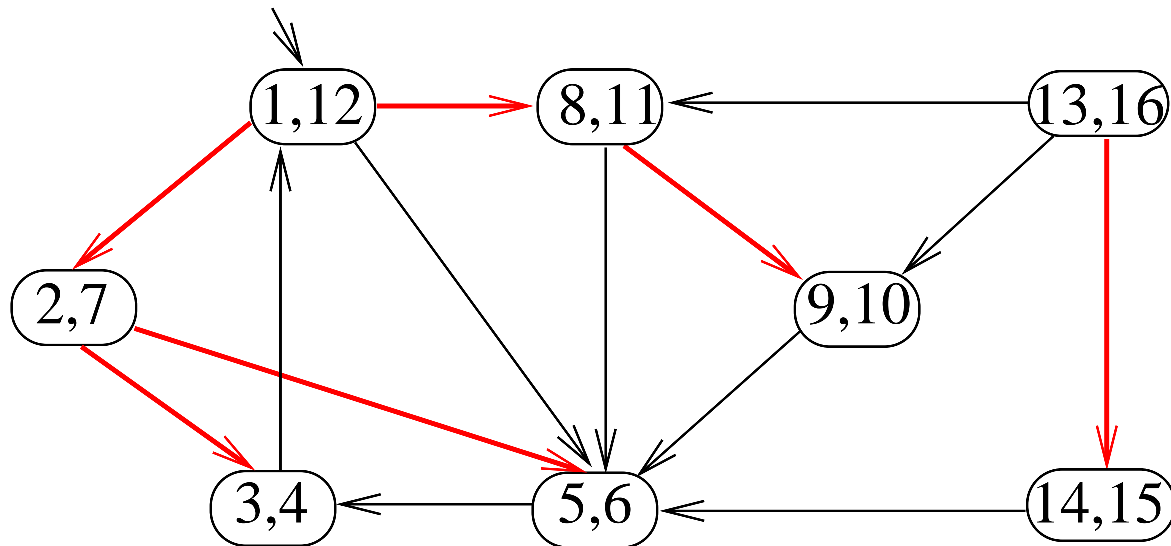

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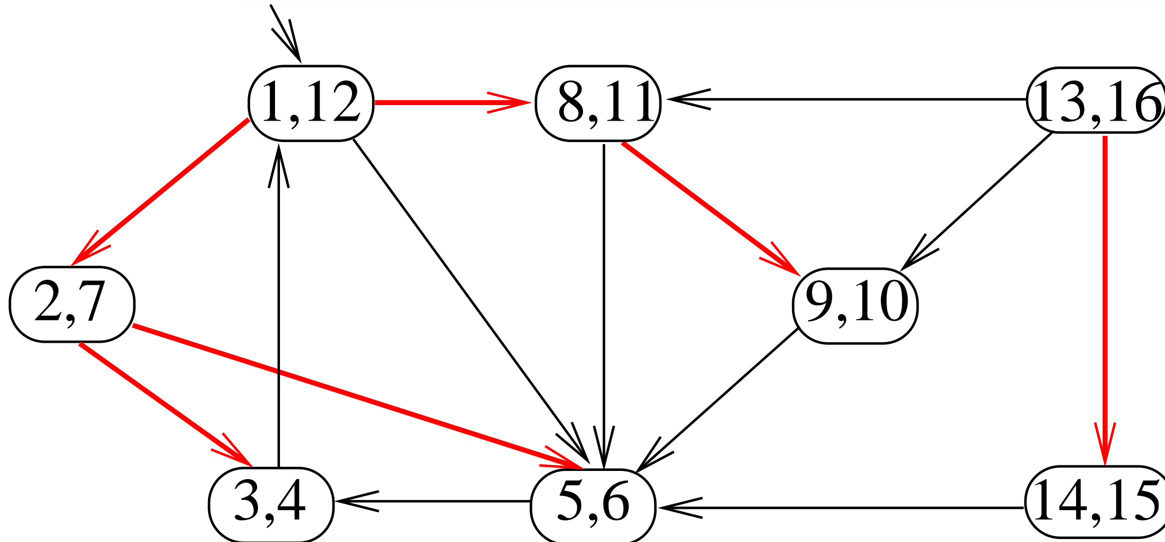


Edge classification:

- TREE edge (T), part of DFS forest
- FORWARD edge (F), from a node to a non-child descendant in DFS tree
- BACK edge (B),leads to an ancestor in DFS tree
- CROSS edge (C), remainder, between trees and subtrees

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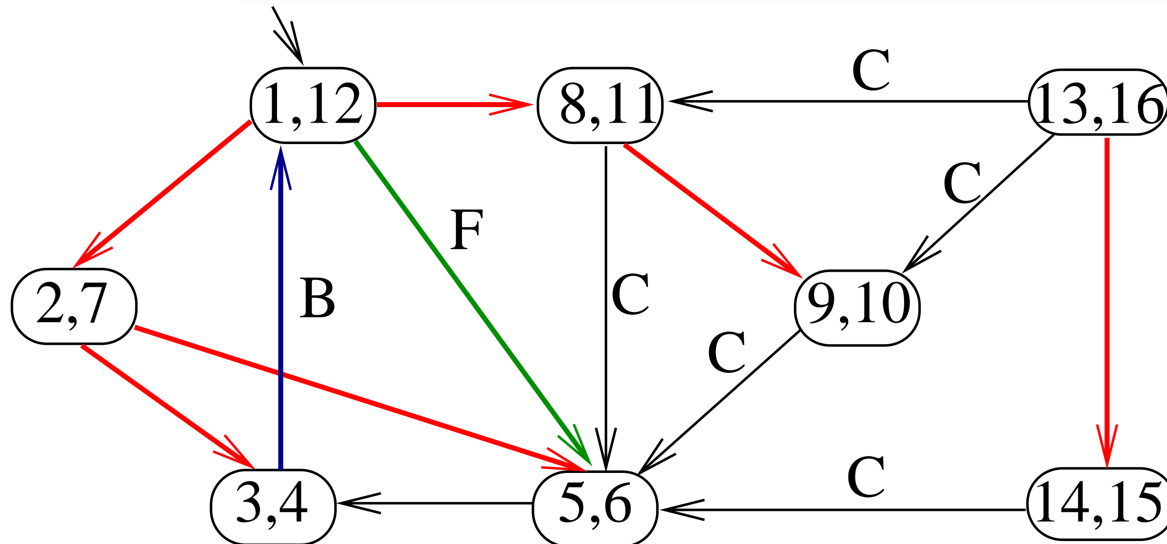


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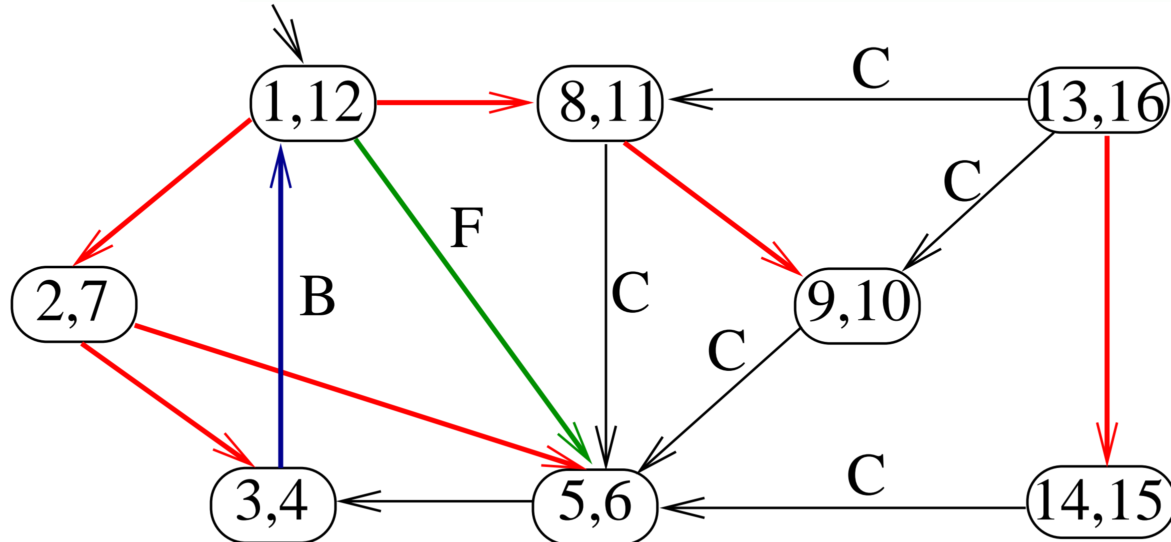


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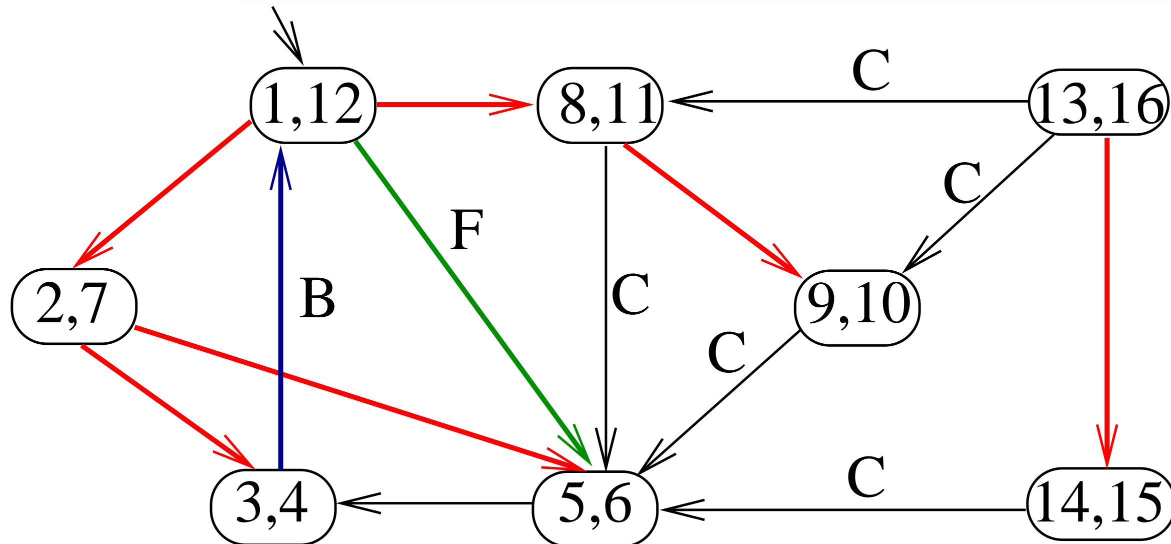


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We can classify edges based on the timestamps ($f[]$, and $d[]$).

# DFS in directed graphs - edge categories

| Type of edge | $d, f$ for $(u, v)$ | Part of DFS-tree? |
|---|---|---|
| Tree | $d[u] < d[v] < f[v] < f[u]$ | yes |
| Forward | $d[u] < d[v] < f[v] < f[u]$ | no |
| Back | $d[v] < d[u] < f[u] < f[v]$ | no |
| Cross | $d[v] < f[v] < d[u] < f[u]$ | no |

# DFS in directed graphs - applications

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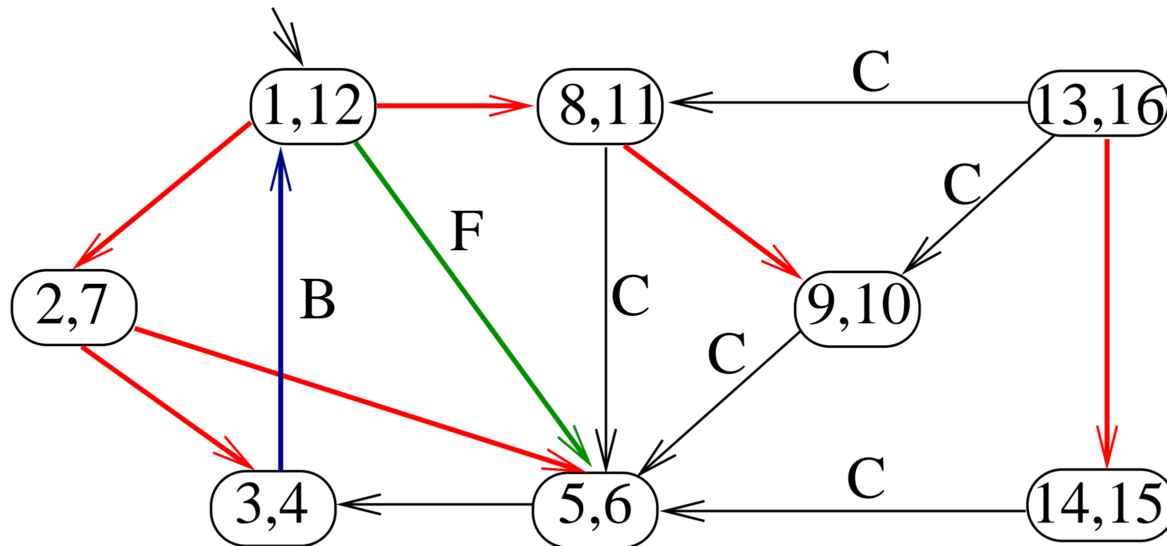

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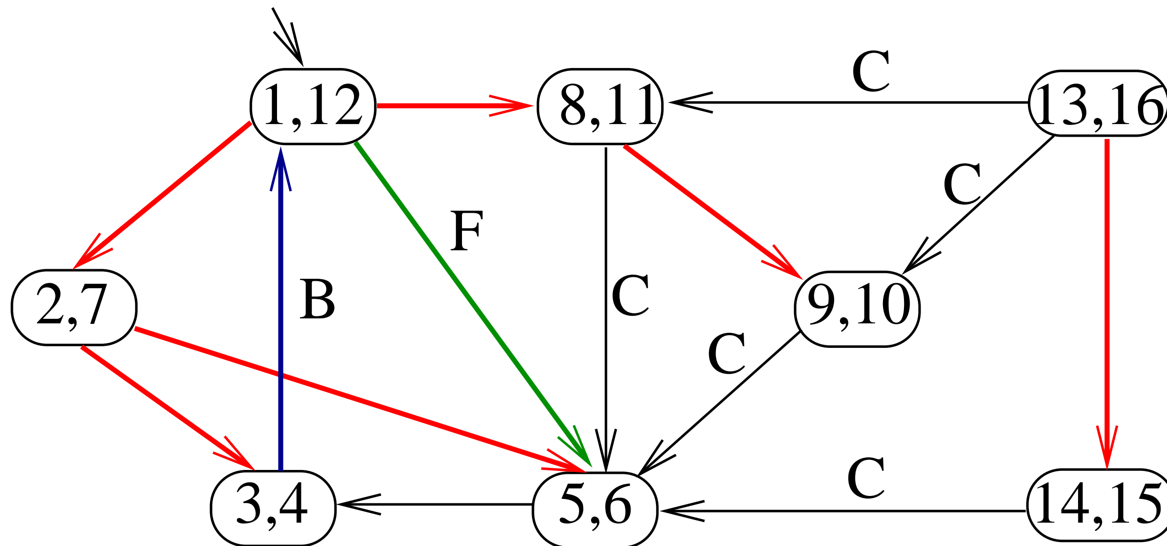


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*Note: the same method works for cycles in undirected graphs, but faster* $O(n)$ *- why ?*

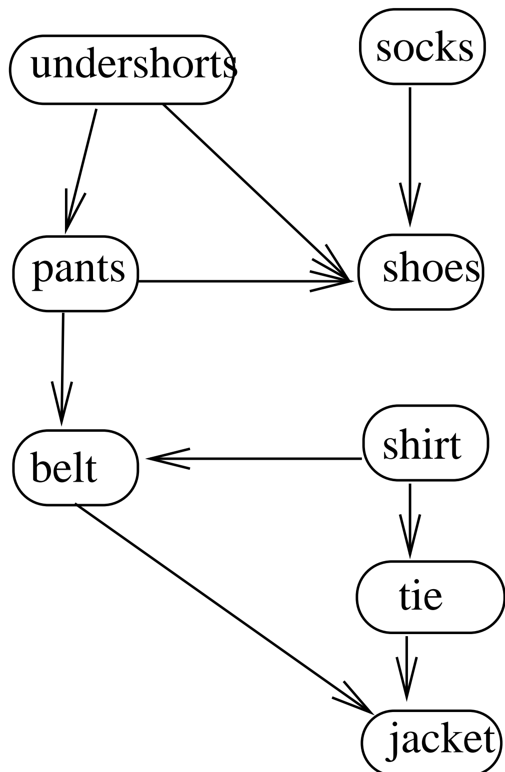
# *Topological Sort of a DAG*

By a topological sorting of a directed acyclic graph we understand a linear ordering of vertices such that if $(u, v) \in E$ then $u \leq v$ in the ordering.

# ***Topological Sort of a DAG***

Example: *Getting dressed*

# Topological Sort of a DAG



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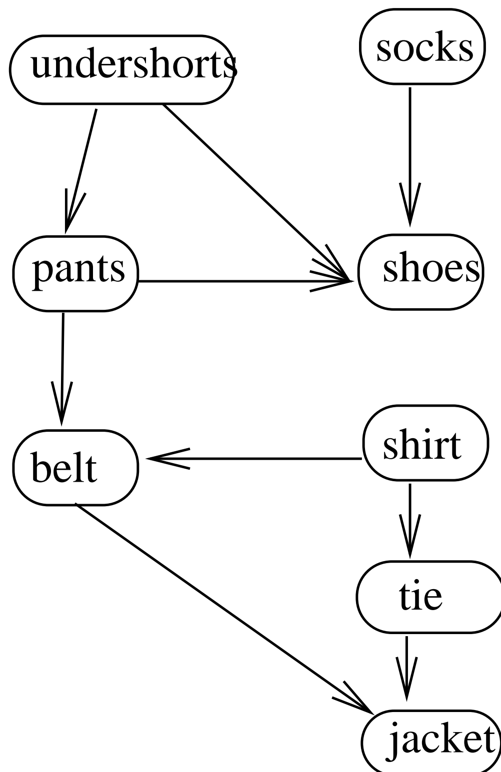


TOPOLOGICAL SORT$(G)$

1. Run DFS$(G)$
2. as a vertex $u$ is finished ($f[u]$ computed), output it in front
3. return the list

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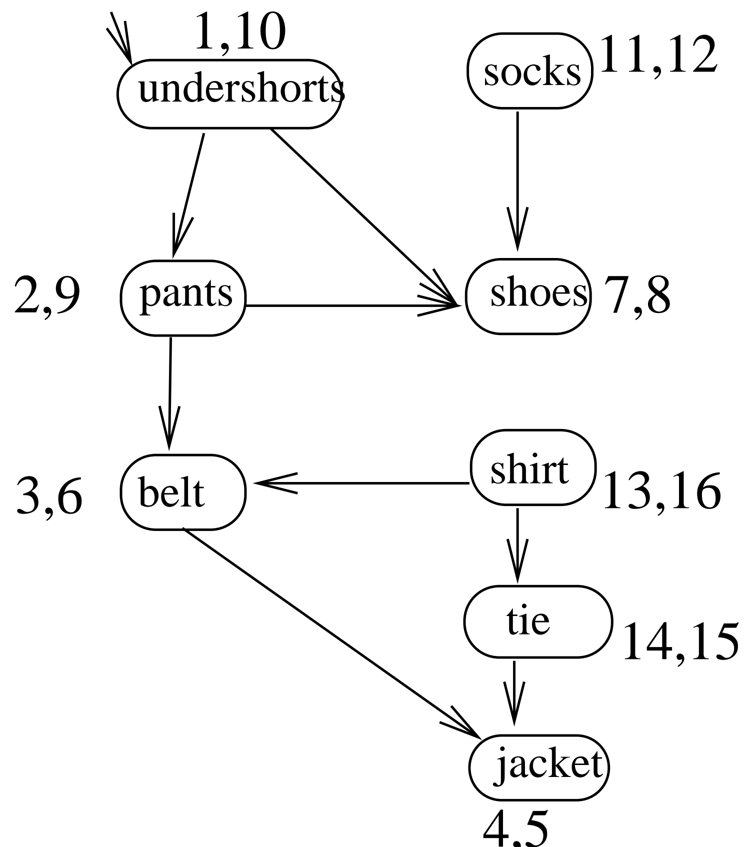


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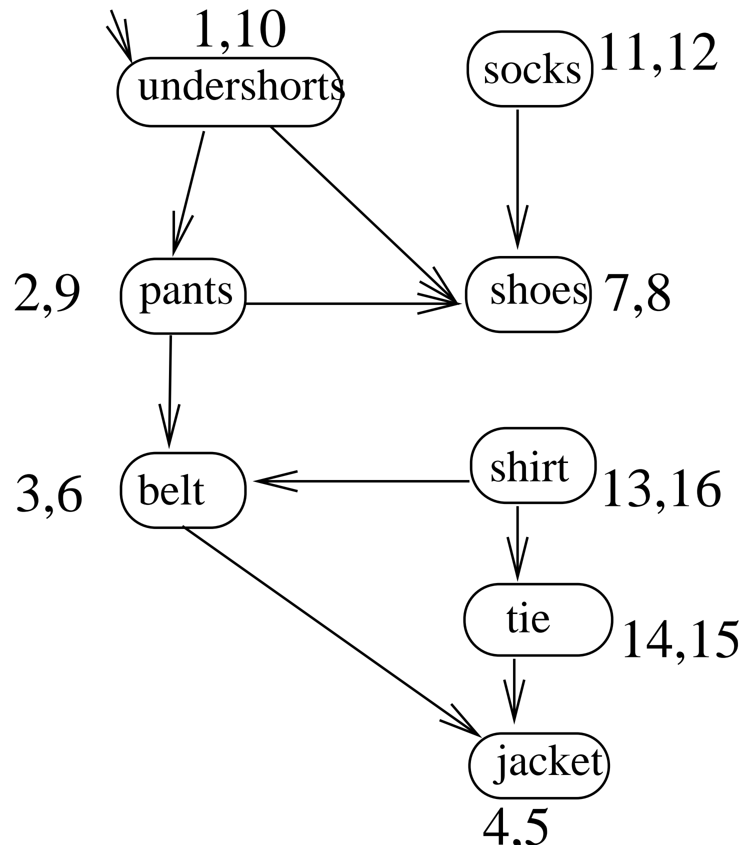


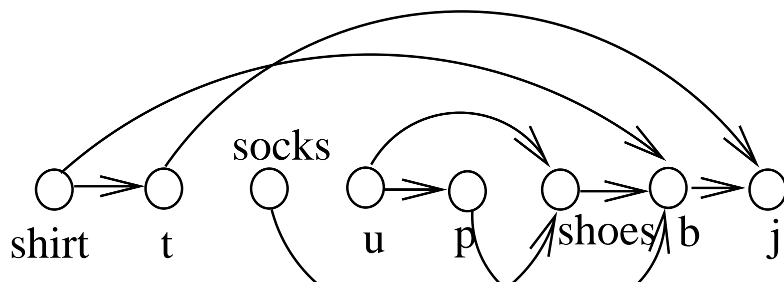


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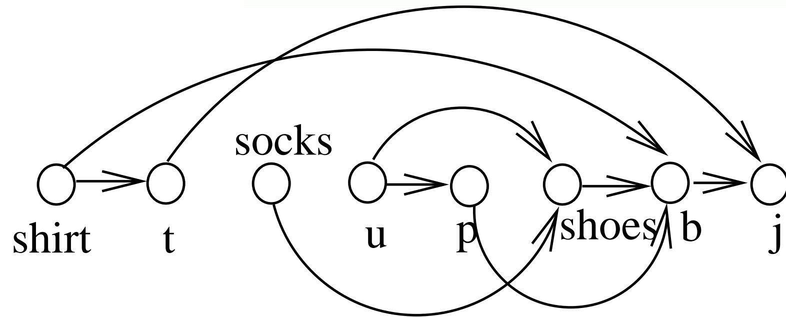

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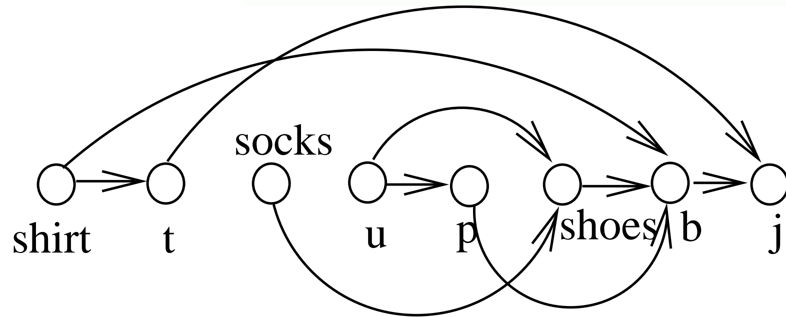


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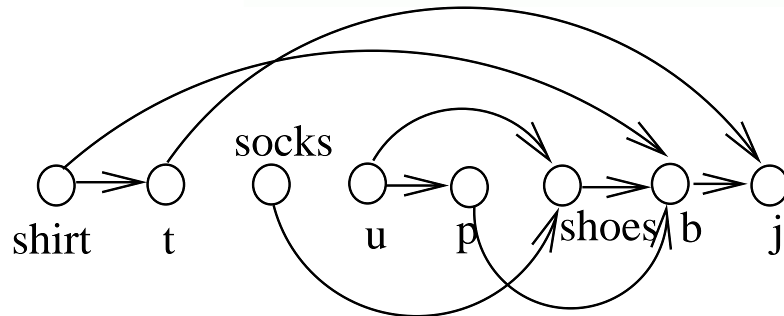


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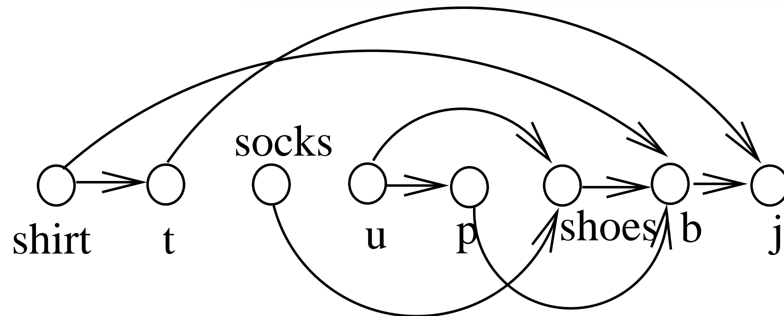


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it is enough to show that *for every pair* $u, v$, *if there is an edge* $(u, v) \in E$ *then* $f[u] > f[v]$.
Proof: the only $(u, v)$ edge such that $f[u] < f[v]$ is a BACK edge but $G$ is a DAG and has no B edges.

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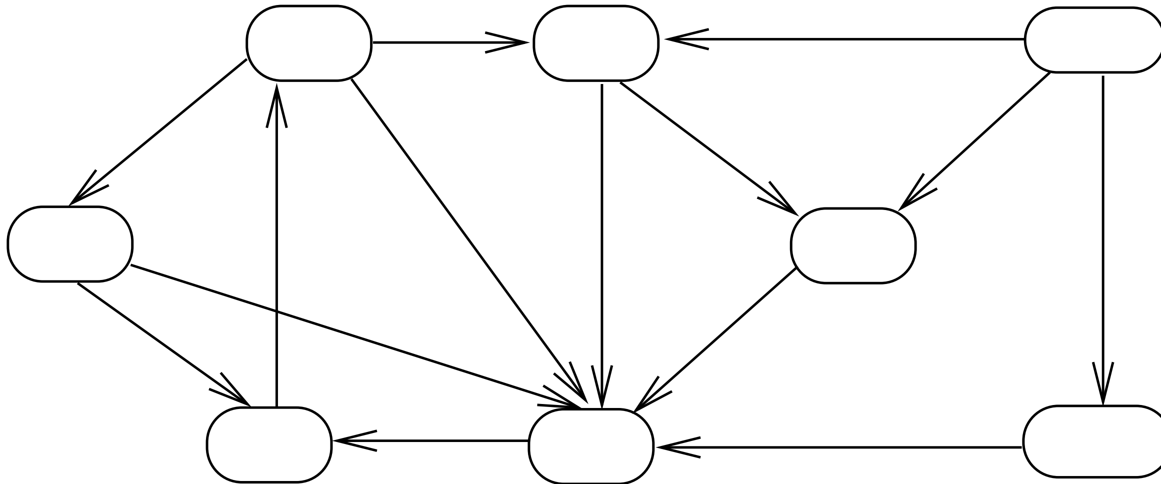

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**Motivation:** searching for *mutual recursions* in program debugging.

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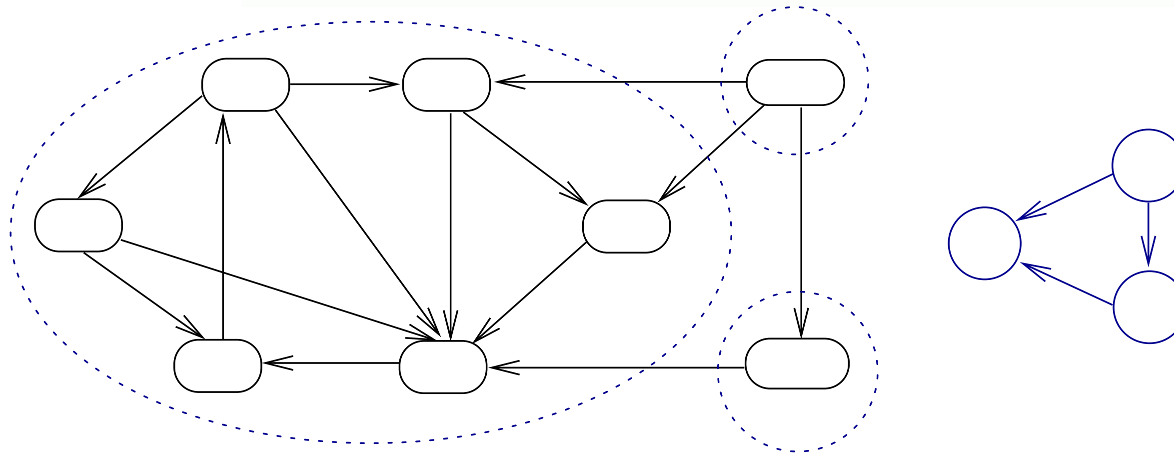

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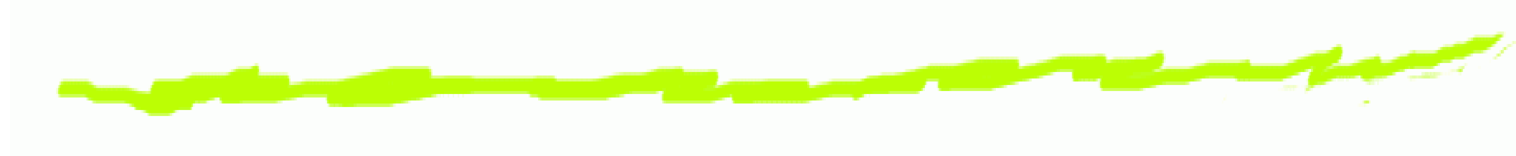

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Follows from the following:
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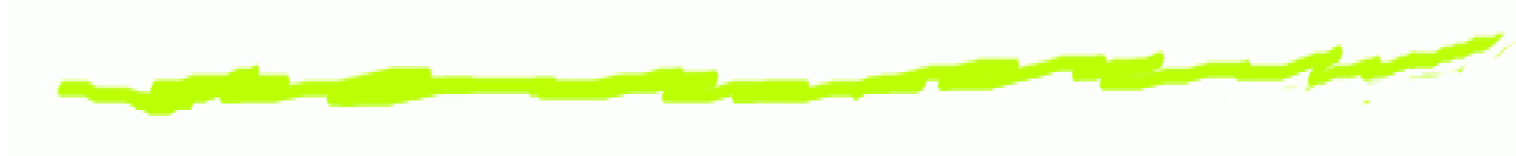
Fact 2 (restated): *the strongly connected components can be topologically sorted by arranging them in decreasing order of their highest finishing time.*

# An efficient algorithm for strong connectivity:



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**Running time:**
computing $G^R$: $O(m+n)$
DFS($G^R$): $O(m+n)$
DFS($G$): $O(m+n)$
**total: :** $O(m+n)$