Design and analysis of algorithms
Lecture 18 & 19

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Minimum spanning tree (MST)

Problem: given a set of computers and a maintenance cost associated with every potential link, build a communication network which is connected and has the cheapest cost.

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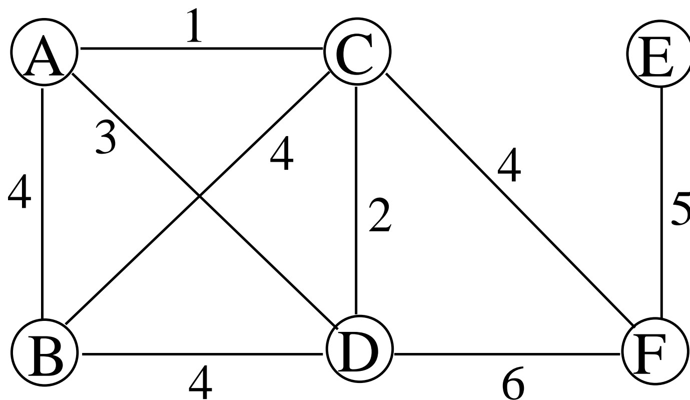

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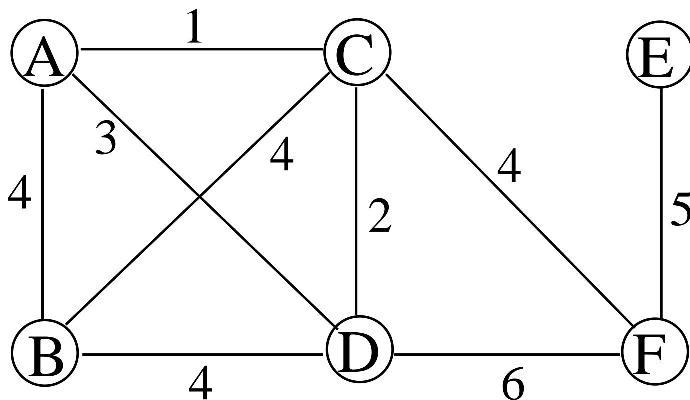

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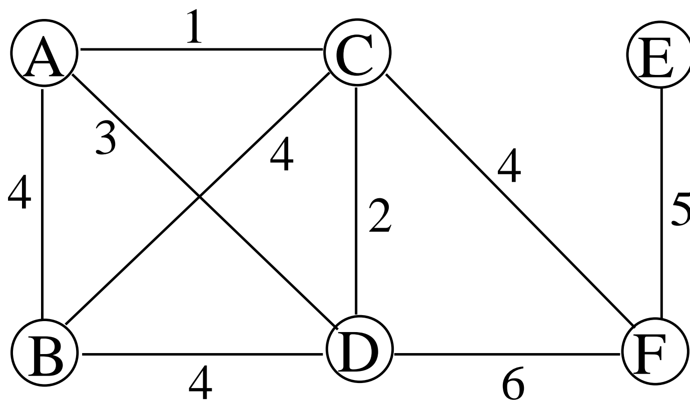


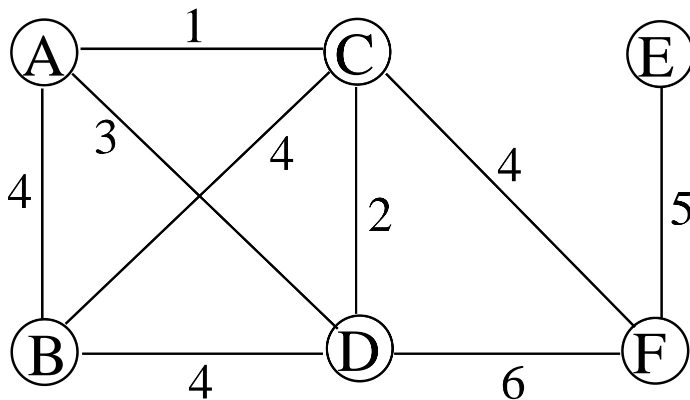
Observation: the optimal set of edges cannot contain a cycle.
Thus, the solution must be connected and acyclic - `TREE`.

Minimum spanning tree (MST)

INPUT: an undirected, connected graph $G = (V, E)$ *with edge weights* $w(e)$.

OUTPUT: a tree $T = (V, E')$ *spanning all the vertices of* G *and minimizing*

$$weight(T) = \sum_{e \in E'} w(e)$$



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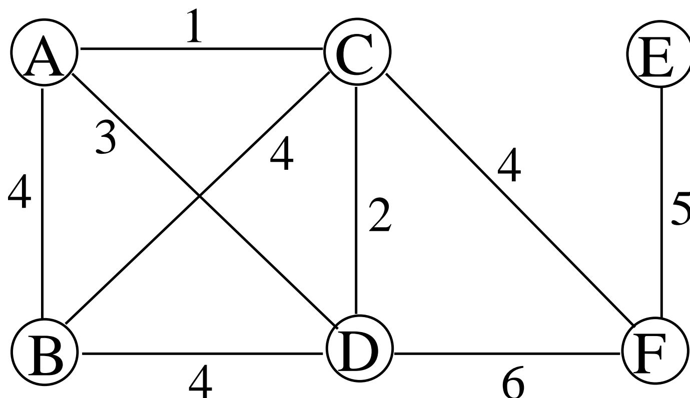

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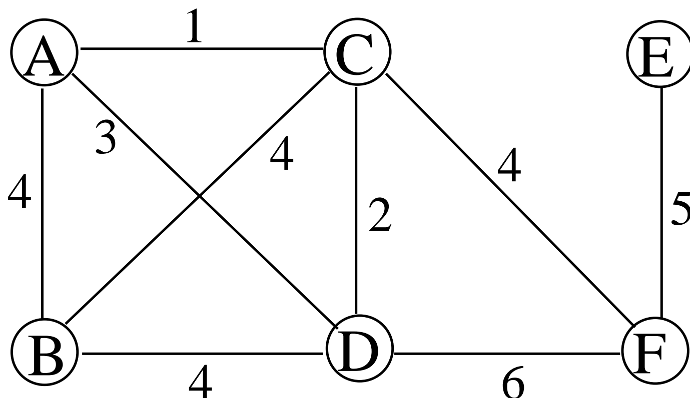

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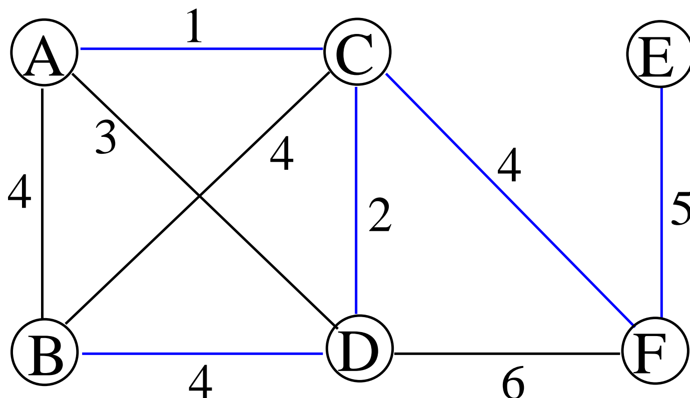

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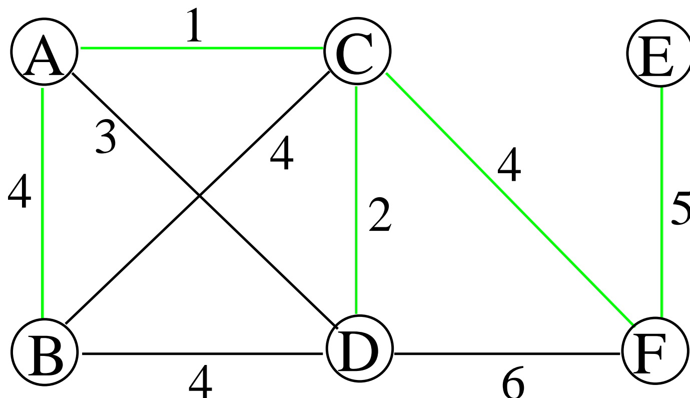

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This a **greedy** method: makes a decision at each step myopically to optimize the underlying criterion.

Why is Kruskal's algorithm correct ?

We will proof a more general property first:

Why is Kruskal's algorithm correct ?

Cut property: *Let* $X \subseteq E$ *be part of some minimum spanning tree* T *of* $G = (V, E)$. *Pick any set of nodes* $S \subset V$ *such that there is no edge in* X *which connects a node in* S *to one in* $V \setminus S$. *If* $e \in E$ *is the minimum-weight edge between* S *and* $V - S$, *then* $X \cup \{e\}$ *is also part of some MST.*

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Proof: Assume $e \notin T$. Then $T \cup \{e\}$ contains a cycle C.
Now C contains another edge e' across the cut $(S, V \setminus S)$.

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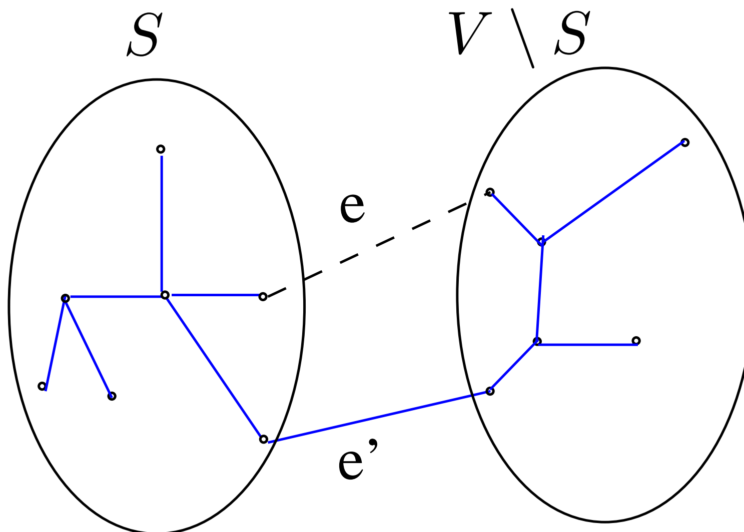

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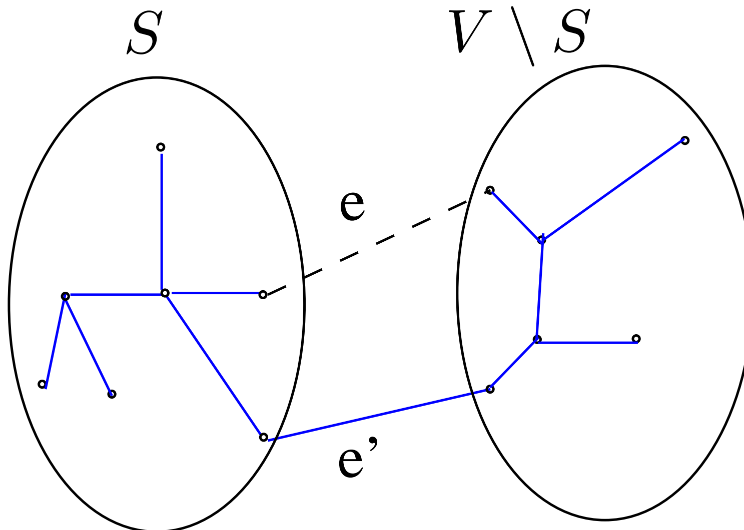

Consider $T' = T \cup \{e\} - \{e''\}$.It is a tree with $w(T') = w(T) + w(e) - w(e') \leq w(T)$, because $w(e) \leq w(e')$. Since T is MST, so is T'.

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In Kruskal's alg.: when we pick an edge connecting two components C and C', it is the lightest edge in the cut $(C, V - S)$ and thus the final tree is optimal, MST.

Implementation of Kruskal's algorithm

Avoiding cycles: every time we want to add edge (u, v) to MST we need to check whether its endpoints u and v are in different connected components.

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Set operations:

- MAKESET(x): creates set $\{x\}$
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KRUSKAL(G, w)
for all $v \in V$ MAKESET(v)
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Total: $O(m \log n)$

b/c all disjoint-set operations take $O(m \log^* n)$ *time, where* $\log^* n$ *is much smaller than* $\log n$.

Prim's greedy choice

builds an MST by growing a single connected component complying with the cut property, r-initial node

Prim's greedy choice

PRIM(G, w, r)
for all $v \in V$ do $cost(v) := \infty$; $prev(v) := nil$
$cost(r) := 0$
$Q := BuilHeap(V)$ {keys = cost values}
while $Q \neq \emptyset$ do

- $u :=$EXTRACT_MIN(Q)
- for each $v \in Adj[u]$ do
  if $v \in Q$ and $cost(v) > w(u, v)$ then
  $cost(v) := w(u, v)$
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Correctness follows from the **Cut Property**.

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- An array:
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  Represent every set by a directed tree, nodes in an arbitrary order, with root as a *representative*

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Example:

Disjoint-sets data structure

- A directed tree:
  for all $x:\ \pi(x)$-parent pointer, $rank(x)$-the height of the subtree rooted at x

  MAKESET(x):

  $\pi(x) := x$

  $rank(x) := 0$

  FIND(x):

  while $x \neq \pi(x)$ do $x = \pi(x)$

  return x

  UNION(x, y): ??

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  return x $\quad O(height(Tree))$

  UNION(x, y): ??