Design and analysis of algorithms Lecture 20& 21

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Disjoint-sets data structure

Set operations:

- MakeSet(x): creates set $\{x\}$
- Find(x): to which set does x belong?
- Union(x, y): merge sets containing x and y.

Disjoint-sets data structure

- An array:
  `Component[s]:=the name of the set containing` s
  Cost:
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Example:

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- A directed tree:
  for all $x:\ \pi(x)$-parent pointer, $rank(x)$-the height of the subtree rooted at x

  MAKESET(x):
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$r_x := Find(x)$

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if $r_x = r_y$ then return

if $rank(r_y) > rank(r_x)$ then

$\pi(r_x) = r_y$

else

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if $rank(r_x) = rank(r_y)$ then

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Properties:

(1) For any x that is not a root, we have $rank(x) < rank(\pi(x))$

(2) any root node of rank k has at least 2^k nodes in its tree

(3) if there are n elements overall, there can be at most $\frac{n}{2^k}$ nodes of rank k

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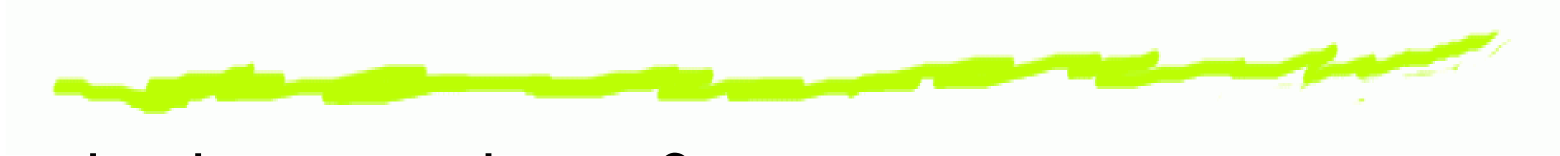
Thus, from (3), the maximum rank is $\log n$ and any disjoint-sets operation takes $O(\log n)$

Path compression



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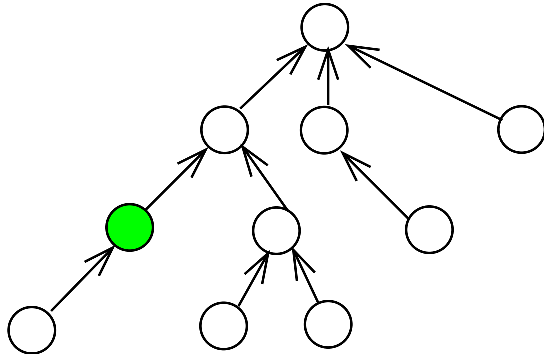

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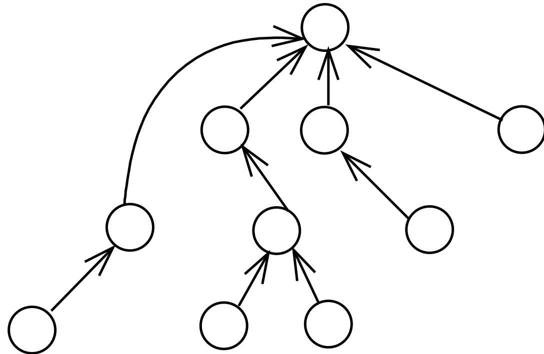

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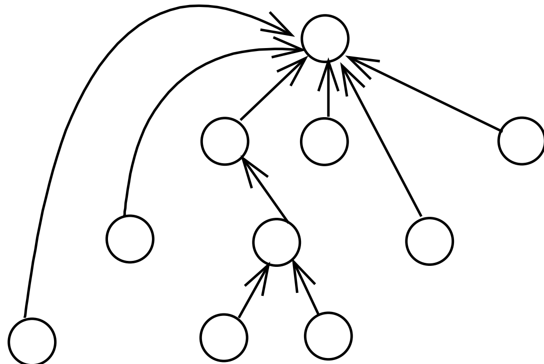

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With this modification later FIND operations on the path will be cheaper.

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Properties (1)-(3) continue to hold.

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We will prove that any sequence of m UNION and FIND operations on n elements take at most $O((m+n)\log^* n)$ steps, where $\log^* n$ is the number of times one must iterate the log function on n before it will result in a number less than or equal to 1.

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To prove it, group the elements according to their ranks in the following way:

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Notice that the number of groups reveals the mysterious $\log^* n$.

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For every x, the cost of FIND(x) is proportional the the number of nodes on the find path from x to the root(before path compression).

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CHILD	(direct child of a root)
GROUP	(x_i s.t $\pi(x_i)$ is in a different group than x_i)
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Thus, the total time is $O(m + n) \log^* n$.