Design and analysis of algorithms Lecture 34 & 35

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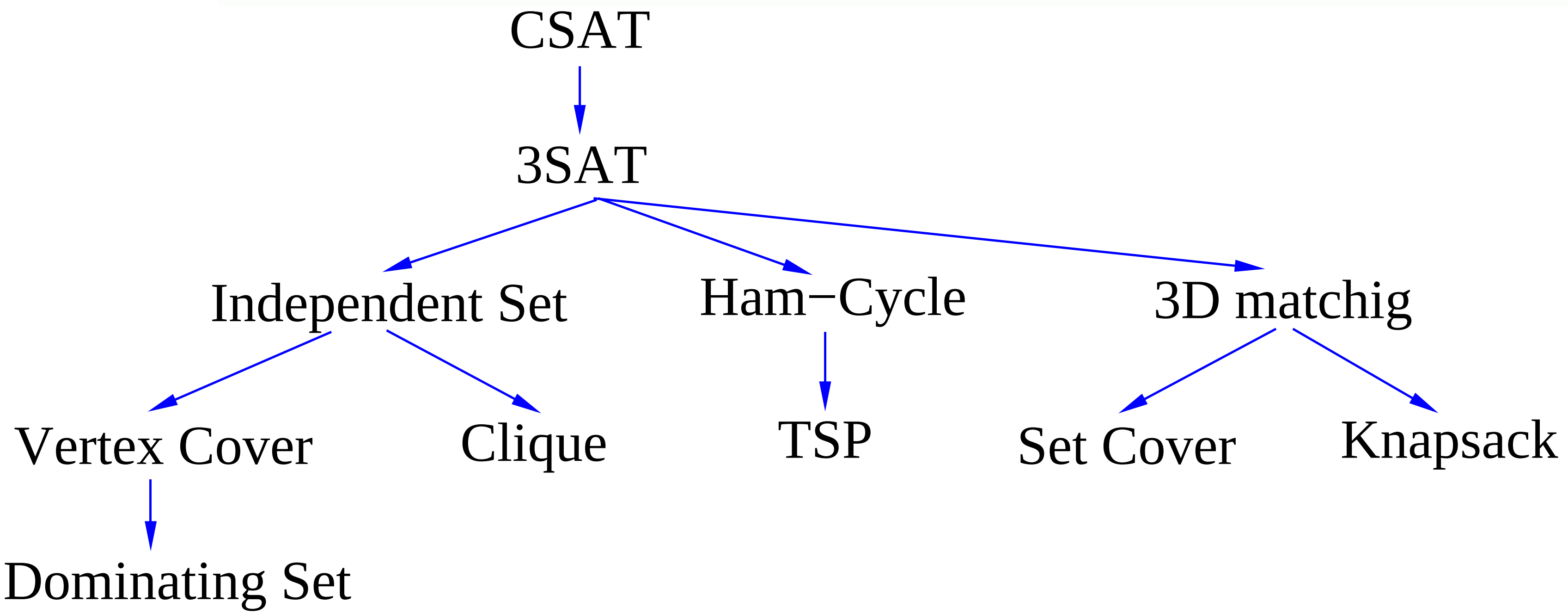
Reductions for NP-completeness



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Easy for paths (your project), trees and bipartite graphs.

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- Will show that G_ϕ has an independent set on m vertices $\Leftrightarrow$ the formula ϕ is satisfiable.

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Graph G_ϕ construction:

- $\forall C \in \phi$ include a triangle in G_ϕ (the vertices in the triangle correspond to the literals of C)
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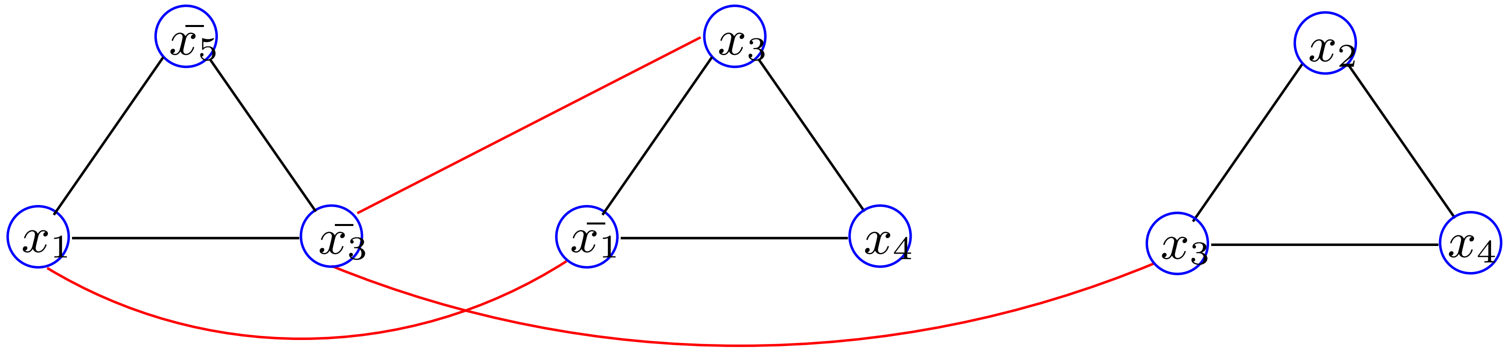

From 3SAT to Independent Set

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if ϕ *is satisfiable, then* $\exists$ *an independent set* I *in* G_ϕ *s.t.* $|I| \geq m$.

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- no two such nodes are adjacent (x and $\bar{x}$ cannot be both satisfied at the same time).

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- For every triangle, we have exactly one vertex in I.
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- This rule yields a satisfying assignment (x and $\bar{x}$ are never together in I).

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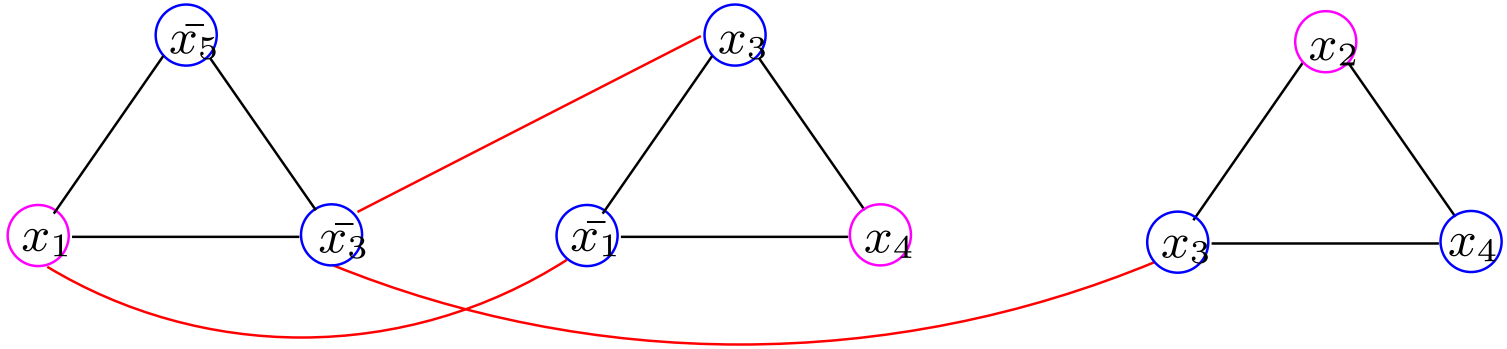

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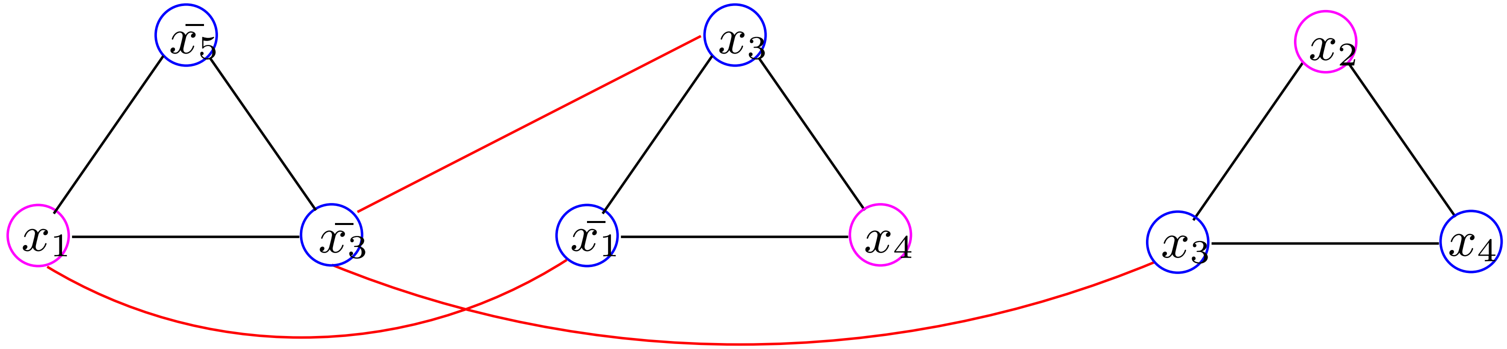

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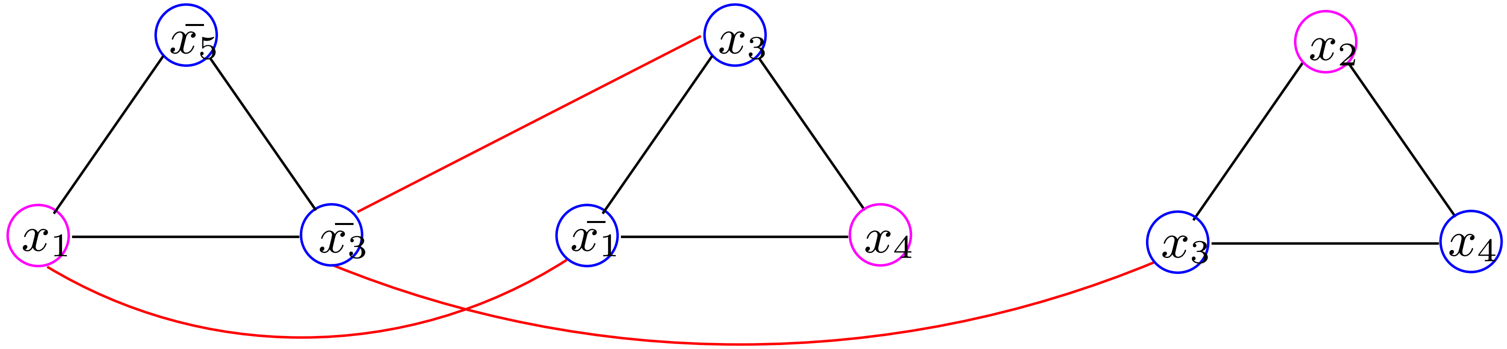

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Satisfying assignment:

$x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0,\ x_5 = 0.$

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   - take k and two graphs: $G = (V, E)$ and $G' = (V, E')$, such that $\forall (u, v) \in G,' \ (u, v) \in E' \Leftrightarrow (u, v) \notin E$. ($G'$ *is a complement of* G.)
   - Every independent set in G is a clique in G' and every clique in G' is an independent set in G.

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4. Thus, G has an independent set I of size at least k if and only if G' has a clique of size k.

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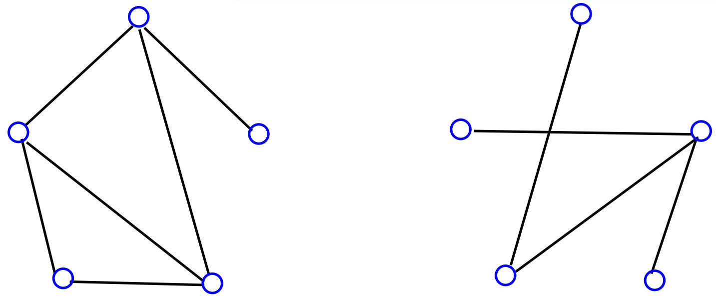

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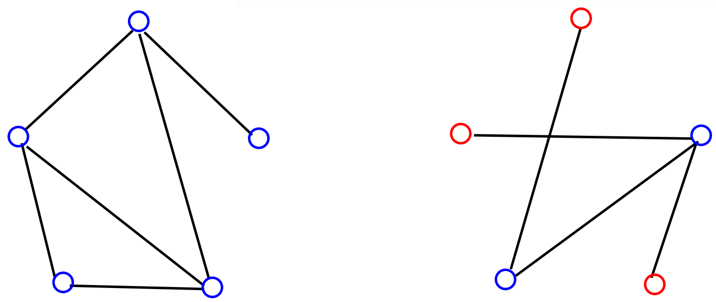

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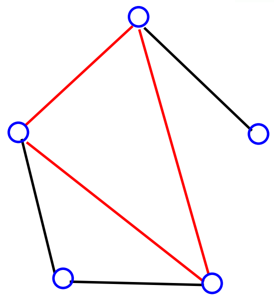

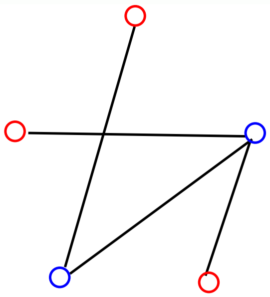

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Lemma *If I is an independent set in a graph $G = (V, E)$, then the set of vertices $C = V \setminus I$ is a vertex cover in G. Furthermore, if C is a vertex cover in G, then $I = V \setminus C$ is an independent set in G.*

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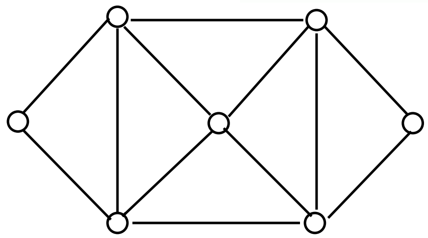

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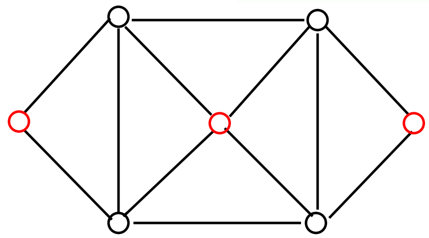

- independent set

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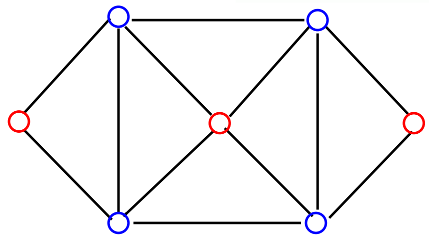

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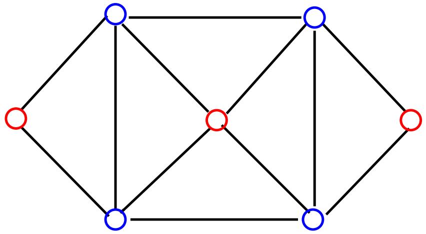

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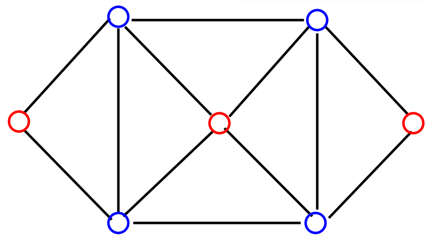

maximum independent set

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Reduction from the Maximum Independent Set:

Take an instance (G, k) of INDEPENDENT SET and produce an instance $(G, n-k)$ of VERTEX COVER.