Divide and Conquer Recurrences

The recurrence:

$$T(n) \;=\; a\,T\frac{n}{b} \;+\; c\,n^d, \quad T(1) \;=\; e$$

where $a > 0$, $b > 1$, $c > 0$, $d \geq 0$ and $e \geq 0$ are constants, has the solution given below:

Case 1: If $d = \log_b a$ then $T(n) \in O(n^d \log n)$.

Case 2: If $d > \log_b a$ then $T(n) \in O(n^d)$.

Case 3: If $d < \log_b a$ then $T(n) \in O(\,n^{\log_b a})$.

Proof: Assume that $n = b^k$. Then, $k = \log_b n$ and $a^k = n^{\log_b a}$.

$$\begin{aligned} T(n) &= a\,T(\frac{n}{b}) + c\,n^d \\ &= a^2\,T(\frac{n}{b^2}) + c\,n^d\,(1 + \frac{a}{b^d}) \\ &\cdots \\ &= a^k\,T(\frac{n}{b^k}) + c\,n^d\,(1 + \frac{a}{b^d} + \cdots + (\frac{a}{b^d})^{k-1}) \end{aligned}$$

Case 1: We have $b^d = a$. Then we get:

$$T(n) \;=\; en^d \;+\; c\,n^d \log_b n.$$

That is, $T(n) \in O(n^d \log n)$.

Case 2: $T(n) = ea^k + c'(1-(\frac{a}{b^d})^k)n^d$, where $c' = (\frac{c}{1\,-\,\frac{a}{b^d}})$. This gives, $T(n) = ea^k + c'\,n^d - c'\,a^k$. Since $d > \log_b a$, n^d dominates a^k. Therefore, we have $T(n) = O(\,n^d)$.

Case 3: $T(n) = ea^k + c'((\frac{a}{b^d})^k - 1)n^d$, where $c' = (\frac{c}{\frac{a}{b^d}\,-\,1})$. This gives, $T(n) = ea^k + c'a^k - c'\,n^d$. Since $d < \log_b a$, a^k dominates n^d. Therefore, we have $T(n) = O(\,a^k) = O(\,n^{\log_b a})$.