\[ Q_j(t+1) = \max\{0, Q_j(t) + A_j(t+1) - 1\} \] 

Renewal Equation

- \( Q_j(t) \) - # of HOL cells at ‘N’ input ports destined for output port ‘j’

Important characteristic of \( Q_j(t) \) is that switching happens just before that and an arrival occurs after that which is called \( A_j(t+1) \).

Thus \( Q_j(t) \) is the queue length at time ‘t’ and \( A_j(t+1) \) represent the cells that arrive after that at time \( t+e \). If the sum of these 2 terms is >0, then there will be a packet switched before \( Q_j(t+1) \) (i.e. a departure).

Therefore,

\[
\text{queue length at } t+1 = (\text{queue length at } t) + (\text{any arrivals}) - \text{(departure)}
\]

![Diagram of queue states and transitions](image_url)

<table>
<thead>
<tr>
<th>N=4</th>
</tr>
</thead>
</table>
| 1            | Q1(t+1) = 1 
| 2            | Q2(t+1) = 0 
| 1            | Q3(t+1) = 0 
| 3            | state at time ‘t+1’ 

\[ t+0.5 \]

\[ t \text{ Arr } t+0.5 \text{ Dep } t+1 \]
At cycle $t+0.5$, the state would be as shown in the figure with cells switched for output ports 1, 2 and 3.

At cycle $t+1.5$

Ideal situation:

For all $j$, $Q_j(t) = 0$. i.e. all the input packets have been switched. If all are going to distinct ports, there will be $N$ arrivals and $N$ departures.

However, it is not possible to obtain ideal throughput because of collisions.

Let the saturation throughput be $\gamma$

Let the expectation of $Q_j(t)$, $E[Q_j(t)] = ?$

There are $N$ HOL cells to start with and $D(t)$ departures at time ‘$t$’.

$Q(t) = N - D(t)$

Therefore, $Q(t)/N = 1 - D(t)/N$

\[ \downarrow \]

$Q(t)/N = 1 - \gamma$

Now $Q(t) = \sum_{j=1}^{N} Q_j(t)$

Therefore, $\left( \sum_{j=1}^{N} Q_j(t) \right)/N = 1 - \gamma$

Calculating the Expectation on each side,
\[ E[ \sum_{j=1}^{N} Q_j(t)/N ] = E[1 - \gamma] \]

Expectation of \(1 - \gamma\) is \(1 - \gamma\) itself.

Every inout port are identical and thus have equal distribution. Therefore,

\[ E[ \sum_{j=1}^{N} Q_j(t)/N ] = \frac{1}{N} \times E[ \sum_{j=1}^{N} Q_j(t) ] \]
\[ = \frac{1}{N} \times N \times E[ Q_j(t) ] \]
\[ = E[ Q_j(t) ] \]

Therefore,
\[ 1 - \gamma = E[ Q_j(t) ] \]

Alternate formula for \(E[ Q_j(t) ]\) for calculating the value of \(\gamma\)

\(D(t) = \# \) departures at time \(t - \varepsilon\)

Consider a fixed \(j\) and when a steady state is reached, \(\lim_{t \to \infty} \lim_{N \to \infty} D(t)/N = \gamma\) which is the saturation throughput.

Therefore, the \(Pr[ A_j(t+1) = k ] = \binom{D(t)}{k} \left( \frac{1}{N} \right)^k \left( 1 - \frac{1}{N} \right)^{D(t)-k} \)

where \(k = 0,1,2,3..D(t)\)
Recall, for a Binomial distribution, 
\[ \lim_{n \to \infty} \lim_{p \to \infty} B(n, p) = \text{Poisson}(\lambda) \]
\[ np \to \lambda \]

Therefore, Considering the Binomial equation Binomial(D(t), 1/N) -> Possion(\( \gamma \))
and rewriting 
\[ \Pr[A_j(t+1) = k] = e^{-\gamma} \left( \frac{\gamma^k}{k!} \right) \]
This means that the number of arrivals at a particular input port ‘j’, \( A_j(t+1) \) is a random variable and follows the Poisson distribution.

Thus, the Renewal equation is a well-defined queueing process. The ‘Arrivals’ follow Poisson Distribution with parameter \( \gamma \), there is one departure. And with this, the average queue size can be determined.

Solving the Renewal equation, the value of \( E[Q_j(t)] \) can be determined and it is
\[ E[Q_j(t)] = \frac{\gamma^2}{2(1-\gamma)} \]

Therefore equating this with \( (1-\gamma) \), we obtain
\[ \frac{\gamma^2}{2(1-\gamma)} = (1-\gamma) \]
\[ \gamma = 2 - \sqrt{2} \]