Algorithm 1: Optical Flow [Horn & Schunk 1980]

Step 0. *k=0.* Initialize all $u_k, v_k = 0$

Step 2. Until some error measure is satisfied do

$$u^k = u_{av}^{k-1} - f_x\frac{P}{D}, v^k = v_{av}^{k-1} - f_y\frac{P}{D}$$

Step 2. If the sum of all E's is sufficiently small, stop, else continue

Algorithm 2: Multiframe Flow

Step 0. *t=0.* Initialize all *u(x,y,0) and v(x,y,0)*

Step 2. for t=1 intil max frames do

$$u(x, y, t) = u_{av}(x, y, t-1) - f_x\frac{P}{D}$$

$$v(x, y, t) = v_{av}(x, y, t-1) - f_y\frac{P}{D}$$

$$\nabla^2 u = u - u_{av}$$
$$\nabla^2 v = v - v_{av} \quad \mathbf{7}$$

we get:

$$(\lambda^2 + f_x^2)u + f_x f_y v = \lambda^2 u_{av} - f_x f_t \quad \mathbf{8}$$

$$(\lambda^2 + f_y^2)v + f_x f_y u = \lambda^2 v_{av} - f_y f_t \quad \mathbf{9}$$

Solving for *u* and *v*:

$$u = u_{av} - f_x \frac{P}{D}$$
$$v = v_{av} - f_y \frac{P}{D} \quad \mathbf{10}$$

where

$$\begin{bmatrix} P = f_x u_{av} + f_y v_{av} + f_t \\ D = f_x^2 + f_y^2 + \lambda^2 \end{bmatrix} \quad \mathbf{11}$$

Guass-Siedel method can be used to solve the above equations using iterative methods.

$$-\frac{\partial f}{\partial} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} \quad (3)$$

$\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are all measurable quantities and $\frac{dx}{dt} = u, \frac{dy}{dt} = v$ are the estimates of the velocities we are looking for in the tow directions.

Using these we can write equation 3 as:

$$-\frac{\partial f}{\partial t} = \nabla f \cdot \boldsymbol{u} = \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v \quad (4)$$

where ∇f is the spatial gradiant of the image.

Now consider a fixed camera with the scene moving past it. The above equations say that the time rate of change of intensity of a point in the image (to first order) is explained as the spatial rate of change in the inensity mutliplied by the velocity that points of the scene move past the camera

The above equation also indicates that the velocity *(u,v)* must lie on a line perpendicular to the vector (f_x, f_y) where f_x and f_y arethe partial derivatives with respect to *x* and *y*. In fact, if the partial derivatives are very accurate then the magnitude of the velocity in the direction (f_x, f_y) is:

$$-\frac{f_t}{\sqrt{(f_x^2 + f_y^2)}} \quad (5)$$

1.2 Calculating Optical Flow by Relaxation

Equation 4 provides a non-unique solution for velocity (*i.e.,* it is not a fully constrained system). In earlier notes on shading methods bt relaxation we discussed a mehod where local smoothness of the surface was used to constrain the system. In a similar way we can assume local smoothness in velocity to use relaxation methods for computing optical flow. We can enforce local smoothness in velocities by using partial derivatives sqaured as error terms, *i.e.,* $u_x^2, u_y^2, v_x^2, v_y^2$.

Using squared-error and minimizing the flow error, we have:

$$E^2(x, y) = (f_x u + f_y v + f_t)^2 + \lambda(u_x^2 + u_y^2 + v_x^2 + v_y^2) \quad (6)$$

Differentiating this with *u* and *v* provides equations for the change in error with respect to *u* and *v*, which must be zero for a minimum. Using

CS7321 Winter 1997: Computer Vision I
Notes on Optical Flow

Irfan Essa

1.0 Optical Flow

Much of the work on computer analysis of visual motion assumes a stationary observer and a stationary background. In contrast, biological systems typically move relatively continiously through the world and the image projected on their retinas varies essentially continiously while they move. Humans perciever smooth continuous motion as such.

Altough biological vision systems are discrete, this quantization is so fine that it is capable of essentially producing continuous outputs. These outputs can mirror the continious flow of the imaged world across the retina. Such continuous information is called *optical flow*. Postulating optical flow as the an input to a perceptual system leads to interesting methods of motion perception.

The optical flow, or instantaneous velocity field assigns to every point on the visual field a two-dimensional "retinal velocity" at which it is moving across the visual field.

In this document we describe how approximations to instantaneous flow may be computed from the usual input situation in a sequence of discrete images.

1.1 The Fundamental Flow Constraint.

An important feature of optical flow is that it can be computed by using just the local information. This is done by modeling motion image as a continuous variation of image intensity $f(x, y, t)$ (OR $I(r, c, t)$) in a Taylor series (*HOT*= Higher Order Terms).

$$f(x + dx, y + dy, t + dt) = f(x, y, t) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial t}dt + HOT \qquad \mathbf{1}$$

Ignoring the higher order terms, we use the following observation:

If indeed the image at some time *t+dt* is the result of the original image at time *t* being moved tranlationally by *dx* and *dy* then, (using first principles of calculas and $dt \to 0$

$$f(x + dx, y + dy, t + dt) = f(x, y, t) \qquad \mathbf{2}$$

From Equation 1 and 2: