Project proposal ‘Convergence of local interactions in catalan structures’

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1 Catalan structures

Catalan structure is a colloquial name given to any possible ‘combinatorial structure’ whose counting sequence is the sequence of Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$. Perhaps the best source for such combinatorial structures is contained in the famous Stanley’s classical list [10]. Local algorithms to sample the uniform distribution over such structures has been proposed in a variety of cases. A classical example is the case of triangulations of the n-gon, in which the dynamics applied consists in swapping (in the only possible way) the common edge of two adjacent triangles. This dynamics will be equivalent to ‘rotating’ an internal vertex of a binary tree with n vertices, which is another well known catalan structure. Dyck paths (up-right paths on a square grid, which stay below the southwest-northeast diagonal), and nonintersecting chords over n vertices positioned on a circle, are other other examples of well studied Catalan structures. Our efforts are concentrated on the following objectives.

1.1 *Triangulation walk*

The ‘edge swapping’ or *triangulation walk* sampling algorithm previously mentioned has been an open problem in the probability community for quite a long time, but still tight analysis of the mixing time is to be seen. Up to date the best lower bound ($\Omega\left(n^{3/2}\right)$ steps), was proved by Molloy, Reed and Steiger [8], who analyzed the time at which the central triangle (containing the largest chord) mixes. On the other hand, in spite of the development of a great source of techniques developed in the last 20 years to estimate the mixing time of Markov chains over combinatorial structures, since the result by McShine and Tetali [7] in which they showed that the relaxation time is upper bounded by $\mathrm{O}\left(n^4\right)$, there has not been significant improvement on the estimates. This problem is posed also in the list of Aldous’ open problems [1] in which he conjectures that the relaxation time for the triangulation walk (and also a similar chain over ‘cladograms’), should be of the order $n^{3/2}$.

Our objective is to study the possibility of improvements at either the lower or the upper bounds. For this, in the case of the lower bound, we want to analyze the ‘height’ statistics for the equivalent walk over binary trees and estimate the necesary steps for this to mix. In the case of the upper bound, there are two approaches to consider: either through the analysis of a modified block dynamics in the sense of [2], or through the comparison with a fastest catalan structure that can be simulated with the triangulation random walk. The analysis of these ‘fast’ catalan structures is the content of the next subsection.

1.2 *Interchange process on Dyck paths*

A catalan path or 'Dyck path' can be described as a sequence of n zeros and n ones such that the number of zeros, counting left to right, always dominates the number of ones. The resulting uniform measure over such paths has been well studied, and a well known result is that its limiting distribution properly scaled coincides with the distribution of the positive Brownian excursion bridge (see [11], [6], [3]).

We choose Dyck paths as canonical structure, with the final objective of translate results concerning mixing of dynamics over Dyck paths to other kind of Catalan structures. A starting point in this analysis is the study of the restricted interchange process over Dick paths, meaning that we choose randomly a transposition (ij) on $[2n]$ and swap the elements at those positions if the resulting path is still a Dick path. If the transpositions are restricted to being adjacent, then the analysis is not difficult, and either path coupling techniques ([5], [4]) or methods from representation theory can be used to study this chain. In the case of general transpositions, the situation seems more challenging, because, even though we expect the chain to mix faster, the points in a path for which the number of zeros is almost the number of ones, act as 'sinks' which bottleneck the dynamics. However, analysis is tractable, and path coupling techniques seem promising to analyze the dynamics. Besides this, there are two other approaches worth trying that fits properly for this chain, namely entropy methods as used by Morris to analyze Monte shuffles [9], and Fourier methods as used by Wilson to analyze lattice paths [12].

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