Towards Faster Integer Programming

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1 Introduction

The Integer Programming (IP) Problem, i.e. the problem of deciding whether a polytope contains an integer point, is a classic problem in Operations Research and Computer Science. Algorithms for IP were first developed in the 1950s when Gomory [Gom58] gave a finite cutting plane algorithm to solve general (Mixed)-Integer Programs. However, the first algorithms with complexity guarantees (i.e. better than finiteness) came much later. The first such algorithm was the breakthrough result of Lentra [Len83], which gave the first fixed dimension polynomial time algorithm for IP. Lenstra's approach revolved on finding "flat" integer directions of a polytope, and achieved a leading complexity term of $2^{O(n^3)}$ where n is the number of variables. Lenstra's approach was generalized and substantially improved upon by Kannan [Kan87], who decreased the complexity to $O(n^{2.5})^n$. Recently, Dadush et al [DPV11] improved this complexity to $\tilde{O}(n^{\frac{4}{3}})^n$ by using a solver for the Shortest Vector Problem (SVP) in general norms. Following the works of Lenstra and Kannan, fixed dimension polynomial algorithms were discovered for counting the number of integer points in a rational polyhedron [Bar94], parametric integer programming [Kan90, ES08], and integer optimization over quasi-convex polynomials [HK10]. However, over the last twenty years the known algorithmic complexity of IP has only modestly decreased. A central open problem in the area therefore remains the following:

Question: Does there exist a $2^{O(n)}$-time algorithm for Integer Programming?

2 Research Plan

The focus of this research is to develop new approaches and tools to tackle this question. We present the general ideas used to solve IP in the past and briefly explain our plan to extend these methods.

Lenstra's Approach: In 1983, Lenstra showed a fundamental connection between Integer Programming and the Geometry of Numbers. The key insight Lenstra leveraged in his algorithm, is that polytopes (or more generally convex bodies) not containing integer points must be "flat", i.e. they cannot be too wide in every direction. More precisely, this fact, known as *Kinchine's Flatness Theorem*, states that if $K \subseteq \mathbb{R}^n$ is a convex set not containing integers, then there exists $z \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\mathrm{width}_K(z) = \sup_{x \in K} \langle z, x \rangle - \inf_{x \in K} \langle z, x \rangle \leq f(n) \tag{2.1}$$

where f depends only on the dimension. In words, the above theorem states than any integer free convex body can be covered by at most $\lceil f(n) \rceil + 1$ parallel integral hyperplanes. Lenstra used this theorem to give a recursive solution procedure for IP: roughly speaking, he computes a flatness direction z for the convex body K of interest (if none exist we can declare K integer feasible), and then recursively solves the $n-1$ dimensional IPs indexed by

$$K \cap \{x \in \mathbb{R}^n : \langle x, z\rangle = k\}, \quad \text{for all } k \in \mathbb{Z} \text{ and } \inf_{x\in K}\langle v, x\rangle \leq k \leq \sup_{x \in K}\langle v, x\rangle.$$

The current best bound for $f(n)$ in (2.1) is $f(n) = \tilde{O}(n^{4/3})$ due to Rudelson [Rud00] building on the work of Banaszczyk et al [BLPS99]. Furthermore, it is known that $f(n) = \Omega(n)$ and is conjectured to be $\Theta(n)$. From the bounds on $f(n)$ the runtime analysis of IP is straightforward. For each dimension k, we create at most $\lceil f(k)\rceil + 1$ subproblems of one lower dimension and continue inductively. In total , starting with an n dimensional problem we get a runtime bounded by $O(f(n))^n$. Given that $f(n) = \Omega(n)$, we do not expect to get better than $\Omega(n)^n$ runtimes using the above approach. In Dadush et al [DPV11], an essentially "optimal" Lenstra type algorithm is given, which we achieve by computing an optimal flatness direction for (2.1) in expected $2^{O(n)}$ time. Previous work relied on computing flatness of an ellipsoidal approximation of K, thereby losing $O(n)$ factors in the worst case.

Higher Dimensional Generalization: To overcome the fundamental limits of Lenstra's approach, we propose the systematic study of a higher dimensional generalization of flatness introduced by Kannan and Lovasz. In [KL88], they show that if $K \subseteq \mathbb{R}^n$ is a convex body not containing integer points, then for some $k \in [n]$, there exists a k dimensional linear subspace $W \subseteq \mathbb{R}^n$ such that

$$\left(\frac{\mathrm{vol}(\pi_W(K))}{\det(\pi_W(\mathbb{Z}^n))}\right)^{\frac{1}{k}} \leq n \tag{2.2}$$

where π_W denotes orthogonal projection onto W, and $\det(\pi_W(\mathbb{Z}^n))$ denotes the determinant of the lattice $\pi_W(\mathbb{Z}^n)$ (volume of fundamental parellelipiped). If we force $k = 1$ (i.e. one dimensional projections), we exactly recover the standard notion of flatness in (2.1). In forthcoming work, the author shows that (2.2) implies that (and is essentially equivalent to) the optimal projection W satisfying

$$|\pi_W(K) \cap \pi_W(L)| \leq (2n)^k$$

The above suggests a higher dimensional generalization of Lenstra's algorithm: find a good projection π_W for K, compute $S = \pi_W(K) \cap \pi_W(L)$, and recurse on $K \cap (W + x)$ for each $x \in S$. In forthcoming work, we use this approach to give an $O(n)^n$ time for IP, which matches the optimal conjectured complexity of Lenstra type algorithms. Here we make algorithmic the proof in [KL88] to find a good π_W, and use a method developed in [DPV11] to compute $\pi_W(K) \cap \pi_W(L)$ efficiently.

Let $g(n)$ denote the best possible right hand side in (2.2). As outlined above, we show that given an oracle to find the best projection W, one can solve IP in $O(g(n))^n$ time. This leads us to the following fundamental questions:

1. How do we find an optimal projection π_W?
2. What is the best value for $g(n)$?

For the first question, in upcoming joint work with Daniele Micciancio, we give an algorithm which produces the best k dimensional projection π_W in $k^{O(kn)}$ time. There is still much room for improvement, though even as is there is hope that we can use our algorithm for super constant k to get a better algorithm for IP.

For the second question, the only non-trivial lower bound is $g(n) = \Omega(\log n)$, which was shown in [KL88]. Furthermore, if we restrict to the case where $K = B_2^n$ (the unit euclidean ball), Kannan and Lovasz show that $g(n) = O(\sqrt{n})$. Interestingly, the techniques developed to prove (2.2) are completely different than those developed to prove the strongest flatness bounds for (2.1). Given the large gap between the known upper and lower bounds and the plethora of new techniques available, we believe that significant progress can be made on this question.

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