CONSTRAINT SATISFACTION PROBLEMS WITH GLOBAL CONSTRAINTS

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1. Introduction

In a celebrated result, [Rag08] showed that, assuming Unique Game Conjecture, every Max-CSP problem has a sharp approximation threshold τ that matches the integrality gap of a natural SDP relaxation. Namely, for any $\epsilon > 0$, $\tau - \epsilon$ approximation is polynomial-time achievable while getting a $\tau + \epsilon$ approximation is NP-Hard. [Rag08] and [RS09] also gave optimal rounding schemes to achieve the ratio.

In spite of the success, there are still some well-known and intensively studied problems of which the complexity of approximating the solution is unknown – such as Graph Expansion, Sparsest Cut, Max-Bisection, Balanced Cut, Min-Bisection, Small Set Expansion, to name a few. These problems, despite their very different characteristics, share a common spirit – they can be viewed as constraint satisfaction problems with some additional global constraints.

Even though these problems share some superficial similarity with usual CSPs, previous works have shown that these problems are actually much harder to handle. While all the CSPs can be optimally approximated in a simple and unified way, the techniques that have been used to deal with global constrained CSPs are very different and usually quite involved. Also, limitations to the generalizations of some well-studied algorithms for usual CSPs are known. For instance, Max-Bisection problem, despite being a simple generalization of Max-Cut, conceals extra difficulties against Goemans-Williamson type algorithm – it is known that the optimal solution can be bounded away from 1 while the natural SDP relaxation has value 1 [GMR$^+$11], in contrary to the case of Max Cut.

Here, we propose a systematical way for getting both better approximation ratio and tighter inapproximability bounds for CSPs with global constraints. To achieve this, we consider Lasserre's SDP hierarchy relaxation [Las01] for global constrained CSPs. One supporting evidence for this approach is a celebrated result by Arora, Rao and Vazirani [ARV04], who showed that one can achieve an $O(\sqrt{\log n})$ approximation for the Sparsest Cut and Balanced Separator problem by adding so called "Triangle Inequalities" to a natural SDP relaxation. Since it is known that "Triangle Inequalities" are just a subset of constraints in the second level of Lasserre's SDP hierarchy, it is reasonable to expect that higher levels of the hierarchy would be helpful in the progress of getting better approximation algorithms for those globally constrained CSPs. We show that this is indeed the case for another important problem in this category – Max-Bisection problem. Specifically, we show that the integrality gap of Max-Bisection relaxation gets reduced as we get to the higher level of the SDP hierarchy. As a consequence, we get an improved (also near optimal) algorithm for Max-Bisection. On the hardness side, we can construct a dictatorship test via an SDP gap instance. However, in the case of globally constrained CSPs, it is still unclear how one can translate an c vs s dictatorship test into an s/c hardness of approximation result.

2. Preliminaries

Due to the space limit, we omit the formal definition of globally constrained CSPs. Instead, we give a concrete and typical example – Max-Bisection.

Definition 1 (Max-Bisection). Given a graph $G = (V, E)$, partition the vertices into two equal pieces such that the number of edges that cross the cut is maximized.

3. Algorithm for Max-Bisection

In this section, we present our results for Max-Bisection.

Theorem 2. *Given a graph $G = (V, E)$ with max-bisection value at least $1 - \epsilon$, there exists an algorithm that finds a bisection with value at least $1 - O(\sqrt{\epsilon}) - O(1/d)$. The algorithm is based on rounding the solutions to some d rounds Lasserre's SDP hierarchy relaxation. Further more, this algorithm is near-optimal under UGC.*

Combining the theorem above and the gap instance given in [GMR+11], we can show that the integrality gap of the hierarchy decreases as the number of rounds increase.

The proof of the theorem above consists of two parts: De-correlating and Biased Rounding. Given a set of vectors to the SDP relaxation, the algorithm first reduce the average correlation between the variables by randomly fixing a constant number of variables. The intuition came from the connection between entropy and mutual information:

$$H(X|Y) = H(X) - I(X;Y)$$

We also give a rounding scheme such that the marginal distribution of each variable is preserved. Since the variables behave like independent after the first phase, we can show the balance of the rounded solution is well-concentrated, thus getting a bisection with high value.

On the hardness side, by combining the "de-correlating" idea and the the approach in [Rag08], we can construct "dictatorship tests", a widely used gadget in the field of hardness of approximation, from an SDP gap instance.

4. Future Directions

Over the recent years, several interesting algorithmic results were developed for various globally constrained CSPs [ARV04, FL06, Ye01]. However, the progress is rather slow on the hardness of approximation side. It seems that closing the gap between approximation and inapproximability is somewhat intractable – part of reason being lack of appropriate and unified framework when dealing with these problems. Here, we propose to use Lasserre's SDP hierarchy as the approach. As already mentioned above, evidences have shown that this direction is somewhat hopeful.

Toward closing the gap, we provide a line of steps for us to get better and better understanding of globally constrained CSPs, Lasserre's hierarchy and their connections.

Re-interpret existing algorithms: As mentioned above, the SDP relaxation in [ARV04] is a special case of the Lasserre's SDP hierarchy. However, the algorithm and analysis in [ARV04] are rather involved. Therefore, a natural question is: can we re-interpret the ARV algorithm (or even develop a simpler one) under the framework of Lasserre's SDP relaxation, and give a simpler proof of the performance guarantee?

Getting such result will not only enhance our understanding of the problem, but also give us more chance to get improved algorithms for various problems, as SDP hierachies automatically provide us a machinery to add more and more valid inequalities to make the relaxation tighter.

Hardness for Max-Bisection and other expansion problems: Although Max-Bisection is a very simple generalization of Max-Cut, the approximation guarantees known are quite different [GW95, Ye01, FL06]. However, there is no distinction between these two problems on the hardness side yet. Thus, we wish to either improve the approximation guarantee or get a tighter inapproximability result for Max-Bisection and other related problems.

Optimal inapproximability results for all globally constrained CSPs: Generalize the question above, we hope to get matching approximation and inapproximability results for all globally constrained CSPs, thus further settling the complexity of a large class of optimization problems. We believe that the construction of dictatorship tests would be helpful for getting such results. Of course, much more work still needs to be done in order to translate a dictatorship test into a hardness result, and it could be very possible that the hardness result might be built on some different conjectures (Small Set Expansion Hypothesis [RS10,RST11], for example).

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