Application for ARC Fellowship, Fall 2011
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Title: Tiny Robots: a Resource Allocation Problem.
(Joint work with Waseem Abbas(ECE).)

One way to explore a broad terrain such as a polar ice cap, the surface of Mars, or the bottom of the ocean is to use a network of small robots. Each robot, while tiny, can carry several types of sensors to detect its environment and relay that information to the others. The problem is that each robot needs access to data from all the different types of sensors, but can only personally carry a fraction of the total number and can only communicate in a limited radius. Given a fixed terrain, number of types of sensors, number of sensors per robot, communication radius, and sensing radius, then, we would like to minimize the number of robots such that each point in the terrain is detected by at least one robot and each robot has access to sensor data from every type of sensor. We approach this robot location problem in two steps.

The first stage is to deploy as few robots as possible such that each point in the terrain can be detected by at least one robot. This is fundamentally a computational geometry problem: "Given a region, say a simple convex polygon, what is the minimum number of unit disks that are sufficient for covering it?" While a seemingly elementary question, there appear to be only limited results, particularly on algorithmic solutions. In fact, the existence of polynomial-time algorithms are not known even if the convex polygon is a sqaure. Our initial research has led to a polynomial time constant factor approximation algorithm, but we would like to improve on this. On the other hand, if, instead of covering a region, we want to cover a finite number of points, the problem of finding the minimum disk cover is known to be NP-hard and a PTAS exists [4]. A useful extension of this research is the question of coverings of simple convex polygon by unit discs in which each disc intersects at least 2 others. In our problem, this will correspond to requiring that each robot has a reasonable number of neighbors with which it can communicate. While we are unsure what the complexity of either of these problem is, we would like to investigate possible solutions, particularly with an eye towards reasonable approximations.

The second half of the problem is, given a configuration of robots, is there an assignment of sensors such that all the robots have access to every type of sensor through communication? We would like to provide both an answer to this yes or no question as well as an algorithm to find a solution if it exists. We consider first the practical version of this problem as it was proposed to us. If we allow each robot to carry two types sensors out of five, can we describe the configurations that allow each robot to see all the types of sensors?

We model this question in terms of graphs in which each vertex is a robot and two vertices have an edge if the corresponding two robots are close enough to communicate. We call f an (a,b)-*domatic function* of G if $f : V(G) \to [a]_b$ such that $\cup_{u \in N[v]} f(u) = \{1, 2, ..., a\}$ for every vertex v in G, where $[a]_b$ is the collection of subsets of $\{1, 2, ..., a\}$ of size b, and $N[v] = \{v, u : uv \in E(G)\}$ for every vertex v. Consequently, our question becomes a decision problem for the existence of $(5, 2)$-domatic functions.

So far, our study of this problem has resulted in the following theorem:

If G is a $K_{1,6}$-free graph of minimum degree at least two, then G has a $(5, 2)$-domatic function.

Note that the graph that we constuct from the geometric situation is always $K_{1,6}$-free; it cannot contain $K_{1,6}$ as an induced subgraph. The assumption of minimum degree two is necessary, since every robot v can only access at most $2(\deg(v) + 1)$ types of sensors. This requirement makes it particularly valuable to solve the unit disk cover problem in which every disk intersects at least other two disks.

This notion of (a, b)-domatic functions is a natural extension of a better studied concept in graph theory, that of domatic partitions. The domatic partition problem has been of interest in graph theory for decades, dating back at least as far as 1975 [2]. The domatic number of a graph is the maximum value k such that the graph admits a $(k, 1)$-domatic function, and the image of any $(k, 1)$-domatic function gives a domatic partition. While it is known that determining whether the domatic number of a graph is at least k is NP-complete, several polynomial time algorithms exist for various classes of graphs [1, 3]. Therefore, determining whether (a, b)-domatic functions of a graph exist is likewise NP-complete in general. We can, however, hope to find results for specific classes of graphs, in particular those arising from the disc covering problem. We hope to leverage our theorem for the existence of $(5, 2)$-domatic functions into results for computing (a, b)-domatic functions on certain graphs and use these theoretical results to construct efficient algorotihms.

This work then is of both theoretical and practical interest. While our eventual goal is to merge solutions to these two problems into an efficient algorithm for answering the robot location question for practical reasons, the theoretical foundation en route is of more general interest. Initially, we would like to work with robots carrying two of a total of five sensors both since we have some theoretical foundation already and this situation is of practical interest. If this work proves successful, there are several natural extensions like generalizing the number of sensor types or the number each robot can carry, though which specifically we pursue will be driven by the needs of the application.

References

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