Homework #1 Solutions

Note: please always include the questions in your answers.

1.

(a). (5 points) Prove that if a positive integer p is not a multiple of 5 then p^2 is also not a multiple of 5.

Proof by contradiction
Suppose p^2 is a multiple of 5, $p^2 = 5m$, m is a positive integer.
By Theorem 2 (page 12), p can be written as a unique product of primes, i.e.
$p = p_1p_2\ldots p_n$, i = 1, 2, …, n
Note that p is not a multiple of 5, so $p_i \neq 5$, i = 1, 2, …, **n**.
So p^2 can also be written as a unique product of primes:
$p^2 = p_1p_2\ldots p_np_1p_2\ldots p_n$ where $p_i \neq 5$, i = 1, 2, …, n
This conflicts with the supposition that p^2 is a multiple of 5 since 5 is a prime.
$\therefore$ If a positive integer p is not a multiple of 5, then p^2 is not a multiple of 5, either.

(b). Prove that $\sqrt{5}$ is irrational.

Proof by contradiction
Suppose x = is rational. By the definition of rational number, we have $x = \frac{p}{q} = \sqrt{5}$
where p and q are positive integers, and ***p* and *q* have no common factors**.
$\therefore p = \sqrt{5}\, q$
$p^2 = 5q^2$
p^2 is a multiple of 5. According to the result of **(a)**, p is also a multiple of 5 (because if p is not a multiple of 5, p^2 should not be a multiple of 5, either).
Let $p = 5k$, then $(5k)^2 = 25k^2 = 5q^2$.
$\therefore 5k^2 = q^2$, i.e. q^2 is a multiple of 5.
For the same reason, q is also a multiple of 5.
Therefore, p and q have a common factor 5, which contradicts with our supposition that p and q have no common factors.
$\therefore$ $\sqrt{5}$ is irrational.

2. (10 points) Prove or disprove: if $p_1, p_2, \ldots, p_n$ are $n \geq 2$ prime numbers, $p_1p_2\ldots p_n + 1$ is not always a prime number.

Proof by counterexample.
Let $n = 2$, $p_1 = 3$, $p_2 = 5$,
then $p_1p_2 + 1 = 16$ which is not a prime number.
$\therefore$ If $p_1, p_2, \ldots, p_n$, $p_1p_2\ldots p_n + 1$ is not always a prime number.

3. (15 points) Use the estimate in the text based on the Prime Number Theorem to give approximate values of the following.

By the Prime Number Theorem, for large enough values of n the fraction of numbers between 1 and n that are prime is approximately $1/\ln n$ and there are approximately $n/\ln n$ primes less than n.

(a). The number of primes between 1 and 10^{30}

$$\frac{10^{30}}{\ln 10^{30}}$$
$$\approx \frac{10^{30}}{69.07755279}$$
$$= 1.45 \times 10^{28}$$

(c). The number of 30-digit primes

the number of primes between 10^{30} and 10^{29}

= the number of primes between 1 and 10^{30} - the number of primes between 1 and 10^{29}

$$= 1.45 \times 10^{28} - \frac{10^{29}}{\ln 10^{29}}$$
$$\approx 1.45 \times 10^{28} - 1.50 \times 10^{27}$$
$$= 1.30 \times 10^{28}$$

(d). The percentage of 30-digit numbers that are primes

the percentage of 30-digit numbers that are primes

$$= \frac{1.30 \times 10^{28}}{10^{30} - 10^{29}}$$
$$\approx 0.014$$
$$= 1.4\%$$

4. (20 points) Let n be a positive integer.

(a). Show that if $n = kl$ with $1 \le k \le l < n$, then k ≤ $\sqrt{n}$

Since $k \le l$, multiply k on both sides and you will get

$$k^2 \le kl = n$$

$\therefore\ k \le \sqrt{n}$

$\therefore$ If $n = kl$ with $1 \le k \le l < n$, then $k \le$.

(b). Give an example for which $n = kl$ and $k = \sqrt{n}$

Let $n = 4$, $k = 2$, $l = 2$, then $k = \sqrt{n}$.

(c). Show that if $n \geq 2$ and n is not a prime, then there is a prime p such that $p \leq \sqrt{n}$ and $p|n$.

Since n is not a prime, n can be written as

$n = kl$ where $1 \leq k < n$, $1 \leq l < n$, and suppose $k < l$.

By Theorem 2 (page 12), k has at least one prime factor, say, p such that $p \leq k$ (if k is a prime itself, $p = k$).

According to the result of **(a)**, $p \leq k \leq \sqrt{n}$.

(d). Conclude that if $n \geq 2$ and n has no prime divisors p with $p \leq \sqrt{n}$, then n is a prime.

Conclude by contradiction

If n is not a prime, according to the result of **(c)**, there is a prime divisor p with $p \leq \sqrt{n}$, which contradicts with the supposition that n has no prime divisors p with $p \leq n$

$\therefore$ If $n \geq 2$ and n has no prime divisor p with $p \leq \sqrt{n}$, then n is a prime.

5. Show that min{x, y} + max{x, y} = x + y for all real numbers x and y. *Suggestion*: Split the argument into two cases.

When $x \leq y$, $\min\{x, y\} = x$, $\max\{x, y\} = y$,

$\min\{x, y\} + \max\{x, y\} = x + y$;

when $x > y$, $\min\{x, y\} = y$, $\max\{x, y\} = y$,

$\min\{x, y\} + \max\{x, y\} = y + x = x + y$.

6. (15 points) The following statements about sets are false. For each statement, give an example, i.e. a choice of sets, for which the statement is false. Such examples are called *counterexamples*. They are examples that are counter to, i.e. contrary to, the assertion.

Proof by counterexample

(a). $A \cup B \subseteq A \cap B$ **for all** A, B

Let $A = \{1, 2\}$, $B = \{2, 3\}$,

$A \cup B = \{1, 2, 3\}$, $A \cap B = \{2\}$

$A \cup B \not\subset$

(b). $A \cap \ddot{O} = A$ **for all** A

Let $A = \{1, 2\}$,

$A \cap \ddot{O} = F \neq A$.

(c). $A \cap (B \cup C) = (A \cap B) \cup C$

Let $A = \{1, 2, 3\}$, $B = \{2\}$, $C = \{3, 4\}$

$A \cap (B \cup C) = \{2, 3\}$, $(A \cap B) \cup C = \{2, 3, 4\}$

$\therefore A \cap (B \cup C) \neq (A \cap B) \cup C$

7. (15 points) Let *A* be a set consisting of 12 positive integers with each integer ≤ 200. Show that there are two disjoint subsets *S* and *T* of *A* whose elements sum to the same value.

A is a set of 12 positive integers. The number of distinct subsets of *A* is $2^{12} = 4096$.

Let $A = \{a_1, a_2, \ldots, a_{12}\}$, $a_i \le 200$ and $1 \le i \le 12$.

$\therefore a_1 + a_2 + \ldots + a_{12} \le 2400$, which means the sum of the elements in any subset of *A* is less than or equal to 2400 which is less than 4096.

$\therefore$ By the Pigeon Hole Principle, there are at least 2 subsets, say *S* and *T* whose elements sum to the same value (consider the 2400 possible sum values as holes and 4096 subsets as pigeons).

Let $S = \{ s_1, s_2, \ldots, s_n\}$, $T = \{t_1, t_2, \ldots, t_m\}$, then

$$s_1 + s_2 + \ldots + s_n = t_1 + t_2 + \ldots + t_m$$

In case *S* and *T* are not disjoint, just take out the common elements to *S* and *T*, the resulting sets *S'* and *T'* are disjoint and have the same element sum value.