CS 1155 HW#2 Sample Solutions

1. Prove the distributive law $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

Proof:

We can use Venn diagrams to solve this problem. Alternatively, we can first show $A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C)$, then we show $A\cup(B\cap C)\supseteq(A\cup B)\cap(A\cup C)$.

To show $A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C)$, we consider an element x in $A\cup(B\cap C)$. Then $x\in A$, or $x\in B$ and $x\in C$. If $x\in A$, then $x\in(A\cup B)$ and $x\in(A\cup C)$, therefore $x\in(A\cup B)\cap(A\cup C)$. If $x\in B$ and $x\in C$, then $x\in(A\cup B)$ and $x\in(A\cup C)$, again we have $x\in(A\cup B)\cap(A\cup C)$.

To show $A\cup(B\cap C)\supseteq(A\cup B)\cap(A\cup C)$, we consider an element x in $(A\cup B)\cap(A\cup C)$. Then $x\in(A\cup B)$ and $x\in(A\cup C)$.

From $x\in(A\cup B)$ we know that $x\in A$ or $x\in B$. From $x\in(A\cup C)$ we know that $x\in A$ or $x\in C$. (*)

If $x\in A$, then $x\in A\cup(B\cap C)$. If $x\notin A$, then x must in both B and C for the above statement (*) to be true, in other words, $x\in B\cap C$, hence $x\in A\cup(B\cap C)$.

So in conclusion, $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

2. For each of the following sets, list all elements if the set has fewer than seven elements. Otherwise, list exactly seven elements of the set.

(a) $\{(m, n) \in \mathbb{N}^2 : m = n\}$

Solution: (0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), etc.

(b) $\{(m, n) \in \mathbb{N}^2 : m + n \text{ is prime}\}$

Solution: (0, 2), (0, 3), (1, 1), (1, 2), (2, 3), (2, 5), (3, 4), etc.

(c)

(d) $\{(m, n) \in \mathbb{P}^2 : \min\{m, n\} = 3\}$

Solution: (3, 3), (3, 4), (3, 5), (3, 6), (4, 3), (5, 3), (6, 3), etc.

(e) $\{(m, n) \in \mathbb{P}^2 : \max\{m, n\} = 3\}$

Solution: (1, 3), (2, 3), (3, 3), (3, 1), (3, 2).

(f) $\{(m, n) \in \mathbb{N}^2 : m^2 = n\}$

Solution: (0, 0), (1, 1), (2, 4), (3, 9), (4, 16), (5, 25), (6, 36), etc.

3. For $n\in\mathbb{Z}$, let $f(n)=½[(-1)^n+1]$. The function f is the characteristic function for some subset of $\mathbb{Z}$. Which subset?

Solution:

Let's use A to represent the unknown subset of $\mathbb{Z}$. Recall that the characteristic function of A is

$$\chi_A = \begin{cases} 1 & \text{for} \quad x \in A \\ 0 & \text{for} \quad x \in \mathbf{Z} \setminus A \end{cases} = f(n)$$

If $f(n)=1=½[(-1)^n+1]$, we know that $(-1)^n=1$, then n must be even.

Similarly, if $f(n)=0=½[(-1)^n+1]$, we know that $(-1)^n=-1$, then n must be odd.
Therefore, $A=\{n\in\mathbb{Z} : n \text{ is even}\}$.

4. We define functions mapping $\mathbb{R}$ into $\mathbb{R}$ as follows: $f(x)=x^3-4x$, $g(x)=1/(x^2+1)$, $h(x)=x^4$. Find

(a) $f \circ f$

Solution: $f\circ f(x) = f(x^3-4x)$
$= (x^3-4x)^3-4(x^3-4x)$

(b) $g \circ g$

Solution: $g\circ g(x) = g(1/(x^2+1))$
$= 1/\{[1/(x^2+1)]^2+1\}$

(c) $h \circ g$

Solution: $h\circ g(x) = h(1/(x^2+1))$
$= 1/(x^2+1)^4$
$= (x^2+1)^{-4}$

(d) $g \circ h$

Solution: $g\circ h(x) = g(x^4)$
$= 1/((x^4)^2+1)$
$= 1/(x^8+1)$

(e) $f \circ g \circ h$

Solution: $f\circ g\circ h(x) = f(g\circ h(x))$
$= f(1/(x^8+1))$ *according to problem (d)*
$= [1/(x^8+1)]^3-4[1/(x^8+1)]$
$= (x^8+1)^{-3}-4(x^8+1)^{-1}$

5. Let $\Sigma=\{a, b, c\}$ and let Σ^* be the set of all words w using letters from Σ. Define $L(w)$=length(w) for all $w\in\Sigma^*$.

(a) Calculate $L(w)$ for the words $w_1=cab$, $w_2=ababac$ and $w_3=\lambda$.

Solution:

$L(w_1)=3$
$L(w_2)=6$
$L(w_3)=0$

(b) Is L a one-to-one function? Explain.

Solution: No. For it to be a one-to-one function, one of the conditions L needs to satisfy is that different words map to different values. Clearly there exist distinct words such as a and b, that map to the same image (in this case, 1). Therefore, L is not a one-to-one function.

(c) The function L maps Σ^* into $\mathbb{N}$. Does L map Σ^* onto $\mathbb{N}$? Explain.

Solution: Yes. Because for any number n in $\mathbb{N}$, there is a word $\underbrace{aa\cdots a}_{n}$ that has a length of n.

(d) Find all words w such that $L(w)=2$.

Solution: They are:

aa, *ab*, *ac*, *ba*, *bb*, *bc*, *ca*, *cb*, *cc*.

6. Here are some functions from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$: SUM$(m, n) = m+n$, PROD$(m, n)=m*n$, MAX$(m, n)=\max\{m, n\}$, MIN$(m, n)=\min\{m, n\}$; here * denotes multiplication of integers.

(a) Which of these functions map $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$?

Solution: All of them. For any k in $\mathbb{N}$:

k = SUM$(k, 0)$ = PROD$(k, 1)$ = MAX$(k, 0)$ = MIN$(k, k+1)$

(b) Show that none of these functions are one-to-one.

Solution: Because for any two distinct elements (m, n) and (n, m) in $\mathbb{N} \times \mathbb{N}$ where $m\neq n$, we know that they map to the same image in $\mathbb{N}$ under those four functions.

7. Consider the function $f:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}\times\mathbb{R}$ defined by

$$f(x, y) = (x+y, x-y)$$

This function is invertible. Show that the inverse function is given by

$$f^{-1}(a,b) = (\frac{a+b}{2}, \frac{a-b}{2})$$

for all (a, b) in $\mathbb{R}\times\mathbb{R}$.

Proof:

First we need to show that $f^{-1}(f(x,y)) = (x,y)$:

$$f^{-1}(f(x,y)) = f^{-1}(x+y, x-y) = (\frac{(x+y)+(x-y)}{2}, \frac{(x+y)-(x-y)}{2}) = (x,y)$$

Second, we need to show that $f(f^{-1}(a,b)) = (a,b)$:

$$f(f^{-1}(a,b)) = f(\frac{a+b}{2}, \frac{a-b}{2}) = (\frac{a+b}{2} + \frac{a-b}{2}, \frac{a+b}{2} - \frac{a-b}{2}) = (a,b)$$

8. Let $f:S\rightarrow T$ and $g:T\rightarrow U$ be invertible functions. Show that $g\circ f$ is invertible and that $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$.

Proof:

Since f and g are invertible, f is a one-to-one correspondence between S and T, and g is a one-to-one correspondence between T and U. It is easy to see that $g\circ f$ is a one-to-one correspondence between S and U and therefore invertible.

Furthermore, $f^{-1}:T\rightarrow S$ and $g^{-1}:U\rightarrow T$ exist, hence $f^{-1}\circ g^{-1}:U\rightarrow S$ exists.

For $x\in U$, $(g\circ f)\circ(f^{-1}\circ g^{-1})(x) = g\circ(f\circ f^{-1})\circ g^{-1}(x)=g\circ g^{-1}(x)=x$

For $y\in S$, $(f^{-1}\circ g^{-1}) \circ(g\circ f)(y)= f^{-1}\circ(g^{-1}\circ g) f(y)= f^{-1}\circ f(y)=y$

Therefore, $(g\circ f)^{-1}= f^{-1}\circ g^{-1}$.

9. Let $f : S \rightarrow T$ be an invertible function. Show that f^{-1} is invertible and that $(f^{-1})^{-1}=f$.

Proof:

Since f is a one-to-one correspondence between S and T, the inverse function f^{-1} is a one-to-one correspondence between T and S, therefore, f^{-1} itself is also invertible.

In order to show

$$(f^{-1})^{-1}=f \qquad (\S)$$

we need to show

$$f(f^{-1}(y))=y \text{ and } f^{-1}(f(x))=x \text{ for all } x\in S,\ y\in T \qquad (¥)$$

Since we know that (¥) is true based on the definition of inverse functions, (§) is true.

10. Pigeon-Hole Principle

(a) A sack contains 50 marbles of four different colors. Explain why there are at least 13 marbles of the same color.

Solution: Since we need to partition the set of 50 elements (marbles) into 4 sets (colors), according to the Pigeon-Hole Principle, at least one of the sets (same color) has $\lceil 50/4 \rceil=13$ elements (marbles). That is to say that at least 13 marbles have the same color.

(b) If exactly 8 of the marbles are red, explain why there are at least 14 of the same color.

Solution: If we know that 8 of the marbles are red, then no other marble could be red, and we need to partition the rest (50-8)=42 marbles into the rest (4-1)=3 colors. According to the Pigeon-Hole Principle, at least $\lceil 42/3 \rceil=14$ marbles must have the same color.

11. Suppose that 73 marbles are placed in eight boxes.

(a) Show that some box contains at least 10 marbles.

Solution: Here we need to partition the set of 73 elements (marbles) into 8 sets (boxes). According to the Pigeon-Hole Principle, at least one of the sets (boxes) has $\lceil 73/8 \rceil = 10$ elements (marbles).

(b) Show that if two of the boxes are empty, then some box contains at least 13 marbles.

Solution: If 2 of the boxes are empty, we need to partition the 73 marbles into the rest (8-2)=6 sets. According to the Pigeon-Hole Principle, at least one of the boxes has $\lceil 73/6 \rceil = 13$ marbles.