

### CS1155 Homework 3 Solutions

1. Let  $p_1, p_2, \dots, p_n$  be  $n$  propositions for  $n \geq 2$ . Write a compound proposition that is true when:
- a. more than half of the propositions are true;

Define  $S = \{ p_1, p_2, \dots, p_n \}$ .  
 $P(S)$  is the power set of  $S$ .

One way to simplify this problem is to realize that if you guarantee that more than half of the propositions are true, then you don't care whether the other propositions are true or false. That is to say that by specifying every possible combination of "just" more than half are true, you also cover all of the cases where "many" more than half are true. So you only need to express the cases where "just" more than half are true.

Define  $P_k(S) = \{ I : I \subseteq S \text{ and } |I| = k \}$ .

Express a single compound proposition that is true for each element of  $P_k(S)$ . This is just an AND of each element of  $I$ .

$$\bigwedge_{k=1}^{|I|} I_k$$

This proposition needs to be defined so that if any one set of  $P_k(S)$  is all true, then the proposition is true. This is an OR over every member of  $P_k(S)$  of the above proposition. Determine which value of  $k$  we want (which is  $n/2 + 1$ ). The resulting proposition is therefore

$$\bigvee_{I \in P_{(n/2 + 1)}(S)} \bigwedge_{k=1}^{|I|} I_k$$

- b. less than half of the propositions are true

If less than half of the propositions are true, then MORE than half of the propositions are false. This means we can use a definition similar to the one above, but just using the negation of each  $I_k$  element.

$$\bigvee_{I \in P_{(n/2 + 1)}(S)} \bigwedge_{k=1}^{|I|} (\neg I_k)$$

- c. an odd number of propositions are true

This problem is different from the first two in that you DO care about the state of all the propositions, therefore you can't simplify it the way I did above.

Define  $\text{Odd}(S) = \{ I : I \subseteq S \text{ and } |I| \text{ is odd} \}$

For each of these subsets then, we need each element in a member of  $\text{Odd}(S)$  to be true, and all elements NOT in that member to be false.

Given  $I \subseteq S$ , define  $J = S - I = \{ x : x \in S \text{ and } x \notin I \}$

$$\bigwedge_{k=1}^{|I|} I_k \quad \bigwedge_{k=1}^{|J|} (\neg J_k)$$

This settles the case for a particular set of of propositions. The same step as in a) and b) needs to be done to extend this to all possible sets.

$$\bigvee_{I \in \text{Odd}(S)} \bigwedge_{k=1}^{|I|} I_k \quad \bigwedge_{k=1}^{|J|} (\neg J_k)$$

- d. an even number of propositions are true

This problem is very similar to part c)

Define  $\text{Even}(S) = \{ I : I \subseteq S \text{ and } |I| \text{ is even} \}$

Define  $J = S - I$  as with part c).

The solution then is the same as for C) but over the set  $\text{Even}(S)$  rather than  $\text{Odd}(S)$

$$\bigvee_{I \in \text{Even}(S)} \bigwedge_{k=1}^{|I|} I_k \quad \bigwedge_{k=1}^{|J|} (\neg J_k)$$

2. Prove that if there is a one-to-one, onto mapping from a set  $S$  to a set  $T$ , then there is a one-to-one and onto mapping from  $P(S)$  to  $P(T)$ .

If there is a one-to-one and onto mapping from  $S$  to  $T$ , then  $S$  and  $T$  must have the same cardinality.

Since the size of the power set is determined only by the number of elements in the original set, if  $S$  and  $T$  have the same cardinality, then their power sets also have the same cardinality. This indicates that there CAN be a 1-1 and onto mapping.

The next step is to demonstrate that for a single subset of  $S$ , there is a unique subset of  $T$  that corresponds to it.

Suppose this is not the case. Then there are two different subsets of  $S$  (say  $X$  and  $Y$ ) which both map to the same subset of  $T$ . Since a single element of  $S$  can only map to one element of  $T$ , then all common elements of  $X$  and  $Y$  map to the same element.

We can eliminate these common elements of  $X$  and  $Y$  and the corresponding elements from the subset of  $T$  (say  $Z$ ). We now have two disjoint subsets  $X'$  and  $Y'$  which map to the same subset of  $T$  (call it  $Z'$ ).

In order for this two happen, one element of  $X'$  and an element of  $Y'$  must both map to the same element in  $Z'$ . These elements in  $X'$  and  $Y'$  are different (since the sets are disjoint). But this contradicts that there is a 1:1 and onto mapping from  $S$  to  $T$ . Therefore there can only be a single subset of  $S$  for each subset of  $T$ .

By a similar argument, you can show that there can only be a single subset of  $T$  for any subset of  $S$ . This means there is a 1:1 mapping between the subsets of  $S$  and subsets of  $T$ . Since these sets are of the same cardinality, this is also an onto mapping.

**3. Exercises 2.2 problem 2**

Consider the proposition “if  $x > 0$  then  $x^2 > 0$ .” Here  $x \in \mathbb{R}$ .

a) Give the converse and the contrapositive of the proposition

Converse: if  $x^2 > 0$  then  $x > 0$

Contrapositive: if  $x^2 \leq 0$  then  $x \leq 0$

b) Which of the following are true propositions: The original proposition, its converse, and its contrapositive?

The original proposition and the contrapositive are true.

**4. Exercises 2.2 problem 5**

Suppose that  $p \rightarrow q$  is known to be false. Give the truth values for  $p \vee q$

The only situation where  $p \rightarrow q$  is false is if  $p$  is true and  $q$  is false.

Under those values  $p \vee q$  is true.

**5. Exercises 2.2 problem 18**

Prove or Disprove.

These problems can also be solved using truth tables, but I'm using the logical equivalences and implications to demonstrate them.

a)  $[p \rightarrow (q \rightarrow r)] \Leftrightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$

$$\begin{aligned} & [(p \rightarrow q) \rightarrow (p \rightarrow r)] \\ & [(\neg p \vee q) \rightarrow (\neg p \vee r)] \\ & [\neg(\neg p \vee q) \vee (\neg p \vee r)] \\ & [(p \wedge (\neg q)) \vee (\neg p) \vee r] \\ & [(p \wedge (\neg q)) \vee (\neg p) \vee (\neg p \wedge \neg q) \vee r] \\ & [(p \wedge (\neg q)) \vee (\neg p \wedge \neg q) \vee (\neg p) \vee r] \\ & [((p \vee \neg p) \wedge \neg q) \vee (\neg p) \vee r] \\ & [\neg q \vee (\neg p) \vee r] \\ & [(\neg p) \vee (\neg q \vee r)] \\ & [(\neg p) \vee (q \rightarrow r)] \\ & [p \rightarrow (q \rightarrow r)] \end{aligned}$$

b)  $[p \oplus (q \rightarrow r)] \Leftrightarrow [(p \oplus q) \rightarrow (p \oplus r)]$

This is not true. Let p, q and r all be true.

$(p \oplus q)$  is false.

$(p \oplus r)$  is false, so  $[(p \oplus q) \rightarrow (p \oplus r)]$  is TRUE.

$(q \rightarrow r)$  is true, so  $[p \oplus (q \rightarrow r)]$  is FALSE.

So they are not equivalent.

c)  $[(p \rightarrow q) \rightarrow r] \Leftrightarrow [p \rightarrow (q \rightarrow r)]$

This is a false statement. Let p, q, and r all be false.

$(p \rightarrow q)$  is true.

$(p \rightarrow q) \rightarrow r$  is false.

$(q \rightarrow r)$  is true.

$p \rightarrow (q \rightarrow r)$  is true, so they are not equivalent.

d)  $[(p \leftrightarrow q) \leftrightarrow r] \Leftrightarrow [p \leftrightarrow (q \leftrightarrow r)]$

$$[(p \leftrightarrow q) \leftrightarrow r]$$

$$[(pq \vee \neg p \neg q) \leftrightarrow r]$$

$$[(pq \vee \neg p \neg q)r \vee \neg(pq \vee \neg p \neg q)\neg r]$$

$$[pqr \vee \neg p \neg qr \vee (\neg(pq) \wedge \neg(\neg p \neg q))\neg r]$$

$$pqr \vee \neg p \neg qr \vee ((\neg p \vee \neg q) \wedge (p \vee q))\neg r$$

$$pqr \vee \neg p \neg qr \vee (p \neg p \vee p \neg q \vee \neg p q \vee q \neg q)\neg r$$

$$pqr \vee \neg p \neg qr \vee p \neg q \neg r \vee \neg p q \neg r$$

$$pqr \vee p \neg q \neg r \vee \neg p(\neg qr \vee q \neg r)$$

$$p(qr \vee \neg q \neg r) \vee \neg p((\neg q \vee \neg r) \wedge (q \vee r))$$

$$p(q \leftrightarrow r) \vee \neg p(\neg(qr) \wedge \neg(\neg q \neg r))$$

$$p(q \leftrightarrow r) \vee \neg p(\neg(qr \vee \neg q \neg r))$$

$$p(q \leftrightarrow r) \vee \neg p(\neg(q \leftrightarrow r))$$

$p \leftrightarrow (q \leftrightarrow r)$  so they are equivalent

e)  $(p \oplus q) \oplus r \Leftrightarrow p \oplus (q \oplus r)$   
 $(p \oplus q) \oplus r$   
 $(p \neg q \vee \neg p q) \oplus r$   
 $(p \neg q \vee \neg p q) \neg r \vee \neg(p \neg q \vee \neg p q) r$   
 $(p \neg q \neg r \vee \neg p q \neg r) \vee (\neg(p \neg q) \wedge \neg(\neg p q)) r$   
 $(p \neg q \neg r \vee \neg p q \neg r) \vee ((\neg p \vee q) \wedge (p \vee \neg q)) r$   
 $(p \neg q \neg r \vee \neg p q \neg r) \vee (p \neg p \wedge q \neg q \wedge \neg p \neg q \wedge p q) r$   
 $p \neg q \neg r \vee \neg p q \neg r \vee \neg p \neg q r \vee p q r$   
 $p(q r \vee \neg q \neg r) \vee \neg p(q \neg r \vee \neg q r)$   
 $p \neg(q \oplus r) \vee \neg p(q \oplus r)$   
 $p \oplus (q \oplus r)$  so they are equivalent

6. Exercises 2.2 problem 20

Verify the following logical implications using truth tables.

a) modus tollens, rule 20

P	Q	$[(P \rightarrow Q) \wedge (\neg Q)]$	$\rightarrow$	$\neg P$
1	1	0	1	0
1	0	1	1	0
0	1	0	1	1
0	0	1	1	1

b) disjunctive syllogism, rule 21

P	Q	$[(P \vee Q) \wedge (\neg P)]$	$\rightarrow$	Q
1	1	0	1	1
1	0	1	1	0
0	1	1	1	1
0	0	0	1	0

7. Exercises 2.2 problem 25

Show that if  $P \Leftrightarrow Q$ ,  $Q \Rightarrow R$  and  $R \Leftrightarrow S$ , then  $P \Rightarrow S$ .

If  $P \Leftrightarrow Q$ ,  $Q \Rightarrow R$  and  $R \Leftrightarrow S$ , then  $(P \leftrightarrow Q)$ ,  $(Q \rightarrow R)$ ,  $(R \leftrightarrow S)$  are true  
 $(P \rightarrow Q)$ ,  $(Q \rightarrow P)$ ,  $(Q \rightarrow R)$ ,  $(R \rightarrow S)$ ,  $(S \rightarrow R)$  are true  
 $(P \rightarrow Q)$ ,  $(Q \rightarrow R)$ ,  $(R \rightarrow S)$ ,  $(Q \rightarrow P)$ ,  $(S \rightarrow R)$  are true  
 by transitivity  $(P \rightarrow Q)$ ,  $(Q \rightarrow R)$  means  $(P \rightarrow R)$   
 by transitivity  $(P \rightarrow R)$ ,  $(R \rightarrow S)$  means  $(P \rightarrow S)$   
 which means that  $P \Rightarrow S$ .

**8.** Exercises 2.5 problem 16

Every compound proposition is equivalent to one that uses only the connectives  $\neg$  and  $\vee$ .  
Find propositions logically equivalent to the following but using only the connectives  $\neg$  and  $\vee$ .

$$p \oplus q$$

$$(p \wedge \neg q) \vee (\neg p \wedge q) \text{ by definition of xor}$$

$$\neg(\neg p \vee q) \vee \neg(p \vee \neg q) \text{ by DeMorgan's Law}$$

**9.** Exercises 2.5 problem 17

Show that  $p \vee q$  and  $p \rightarrow q$  are logically equivalent to propositions using only the connectives  $\neg$  and  $\wedge$ .

$$(p \vee q) \Leftrightarrow \neg\neg(p \vee q) \text{ by double negation}$$

$$\Leftrightarrow \neg(\neg p \wedge \neg q) \text{ by DeMorgan laws}$$

$$p \rightarrow q \Leftrightarrow \neg\neg(p \rightarrow q) \text{ by double negation}$$

$$\Leftrightarrow \neg(p \wedge \neg q) \text{ by Logical equivalence 11b}$$

**10.** Exercises 11.3 problem 6

Which of the following sets are countable? Countably infinite?

A  $\{0, 1, 2, 3, 4\}$

B  $\{n \in \mathbb{N}: n \leq 73\}$

C  $\{n \in \mathbb{Z}: n \leq 73\}$

D  $\{n \in \mathbb{Z}: |n| \leq 73\}$

E  $\{5, 10, 15, 20, 25, \dots\}$

Countable: All of them are countable

Countably infinite: Only C and E are countably infinite