CS 1155 HW4 Sample Solutions

1. Exercises 2.5, problem 9 (b) and (c)

Find a logical equivalence in Table 1 on page 84 from which the use of Substitution Rule (a) yields the indicated equivalence.

(b) $[p\vee(q\wedge(r\wedge s))]\Leftrightarrow[(p\vee q)\wedge(p\vee(r\wedge s))]$

comes from distributive law (rule 4a) $[p\vee(q\wedge r)]\Leftrightarrow[(p\vee q)\wedge(p\vee r)]$ with r replaced by $(r\wedge s)$

(c) $\neg[(\neg p\wedge r)\vee(q\rightarrow r)]\Leftrightarrow[\neg(\neg p\wedge r)\wedge\neg(q\rightarrow r)]$

comes from DeMorgan law (rule 8a) $\neg(p\vee q)\Leftrightarrow(\neg p\wedge\neg q)$ with p replaced by $(\neg p\wedge r)$ and q replaced by $(q\rightarrow r)$

2. Exercises 2.5, problem 10 (a), (b), and (c)

Find a logical implication in Table 2 on page 86 from which Substitution Rule (a) yields the indicated implication.

(a) $[\neg p\vee q]\Rightarrow[q\rightarrow((\neg p\vee q)\wedge q)]$

comes from rule 22 $p\Rightarrow[q\rightarrow(p\wedge q)]$ with p replaced by $(\neg p\vee q)$

(b) $[(p\rightarrow q)\wedge(r\rightarrow q)]\Rightarrow[(\neg q\vee\neg q)\rightarrow(\neg p\vee\neg r)]$

comes from destructive dilemma (rule 27a)
$[(p\rightarrow q)\wedge(r\rightarrow s)]\Rightarrow[(\neg q\vee\neg s)\rightarrow(\neg p\vee\neg r)]$ with s replaced by q

(c) $[((p\rightarrow s)\rightarrow(q\wedge s))\wedge\neg(q\wedge s)]\Rightarrow\neg(p\rightarrow s)$

comes from modus tollens (rule 20) $[(p\rightarrow q)\wedge\neg q]\Rightarrow\neg p$ with p replaced by $(p\rightarrow s)$ and q replaced by $(q\wedge s)$

3. Exercises 2.5, problem 18 (a), (b), (c), (d), and (e)

(a) Show that $\neg p \Leftrightarrow p|p$

can prove either using logical equivalences or a truth table

$\neg p$	
$\neg(p \wedge p)$	idempotent law (rule 5b)
$p|p$	Sheffer stroke

p	¬p	⇔	p|p
0	1	**1**	1
1	0	**1**	0

(b) Show that $p \vee q \Leftrightarrow (p|p)|(q|q)$

$p \vee q$	
$(p \wedge p) \vee (q \wedge q)$	idempotent law (rule 5b)
$\neg[\neg(p \wedge p) \wedge \neg(q \wedge q)]$	DeMorgan law (rule 8c)
$\neg[(p|p) \wedge (q|q)]$	Sheffer stroke
$(p|p)|(q|q)$	Sheffer stroke

p	q	p∨q	⇔	(p|p)	|	(q|q)
0	0	0	**1**	1	0	1
0	1	1	**1**	1	1	0
1	0	1	**1**	0	1	1
1	1	1	**1**	0	1	0

(c) Find the proposition equivalent to $p \wedge q$ using only the Sheffer stroke

$p \wedge q$	
$\neg\neg(p \wedge q)$	double negation
$\neg(p|q)$	Sheffer stroke
$(p|q)|(p|q)$	(a)

(d) Do the same for $p \rightarrow q$

$p \rightarrow q$	
$\neg(p \wedge \neg q)$	implication (rule 10b)
$\neg(p \wedge (q|q))$	(a)
$p|(q|q)$	Sheffer stroke

(e) Do the same for $p\oplus q$

$p\oplus q$	
$\neg(p\leftrightarrow q)$	equivalence of XOR (exercises 2.2, problem 13)
$\neg[(p\rightarrow q)\wedge(q\rightarrow p)]$	equivalence of biconditional (chapter 2.2, example 2)
$\neg[(p\mid(q\mid q))\wedge(q\mid(p\mid p))]$	(a)
$(p\mid(q\mid q))\mid(q\mid(p\mid p))$	Sheffer stroke

4. Exercises 2.6, problem 13(b)

Give formal proof of the following

(b) $\neg N$ from $H\wedge\neg R$ and $(H\wedge N)\rightarrow R$. Suggestion: proof by contradiction

Proof	Reasons
1. $H\wedge R$	hypothesis
2. H	1; simplification (rule 29)
3. $\neg(\neg N)$	negation of conclusion
4. N	3; double negation
5. $H\wedge N$	2,4; conjunction (rule 34)
6. $(H\wedge N)\rightarrow R$	hypothesis
7. R	5,6; modus ponens (rule 30)
8. $\neg R$	1; simplification (rule 29)
9. $R\wedge\neg R$	7,8; conjunction (rule 34)
10. contradiction	9; rule 7b

5. Exercises 2.6, problem 12 (a), (b), and (c)

For each of the following, give a formal proof of the theorem or show that it is false by exhibiting a suitable row of the truth table.

(a) If $(q\wedge r)\rightarrow p$ and $q\rightarrow\neg r$, then p

false; consider these rows of the truth table

p	q	r	[((q∧r)	→	p)	∧	(q	→	¬r)]	→	p
0	0	0	0	1	0	1	0	1	1	**0**	0
0	0	1	0	1	0	1	0	1	0	**0**	0
0	1	0	0	1	0	1	1	1	1	**0**	0

(b) If $q \vee \neg r$ and $\neg(r \to q) \to \neg p$, then p

false; consider these rows of the truth table

p	q	r	[(q∨¬r)	∧	(¬(r→q)	→	¬p)]	→	p
0	0	0	0 1 1	1	0 1	1	1	**0**	0
0	1	0	1 1 1	1	0 1	1	1	**0**	0

(c) If $p \to (q \vee r)$, $q \to s$ and $r \to \neg p$, then $p \to s$

true

Proof	Reasons
1. $r \to \neg p$	hypothesis
2. $\neg\neg p \to \neg r$	1; contrapositive (rule 9)
3. $p \to \neg r$	2; double negation (rule 1)
4. $p \to (q \vee r)$	hypothesis
5. $(p \to \neg r) \wedge (p \to (q \vee r))$	3,4; conjunction (rule 34)
6. $p \to (\neg r \wedge (q \vee r))$	5; rule 12b
7. $p \to [(\neg r \wedge q) \vee (\neg r \wedge r)]$	6; distributive law (rule 4b)
8. $p \to [(\neg r \wedge q) \vee 0]$	7; rule 7b
9. $p \to [(\neg r \wedge q)$	8; identity law (rule 6a)
10. $(p \to \neg r) \wedge (p \to q)$	9; rule 12b
11. $p \to q$	10; simplification (rule 29)
12. $q \to s$	hypothesis
13. $p \to s$	11,12; hypothetical syllogism (rule 33)

6. Exercises 3.1, problem 6

Consider the relation R on Z defined by $(m,n) \in R$ if and only if $m^3 - n^3 \equiv 0 \pmod 5$. Which of the properties (R), (AR), (S), (AS) and (T) are satisfied by R?

(R), (S), and (T) are satisfied
(AR) is not satisfied [(x,x) would never fail since $x^3 - x^3 = 0 \equiv 0 \pmod 5$]
(AS) is not satisfied [consider (1,6) and (6,1): $1^3 - 6^3 = -215 \equiv 0 \pmod 5$; $6^3 - 1^3 = 215 \equiv 0 \pmod 5$; but $1 \neq 6$]

7. Exercises 3.1, problem 10 (a) and (b)

Give an example of a relation that is

(a) antisymmetric and transitive but not reflexive

$(x,y) \in R$ if $x = \max(y,1)$, R is defined on N

(b) symmetric but not reflexive or transitive

$(x,y) \in R$ if $\max(x,y) = c$, c is a constant, R is defined on N

8. Exercises 3.1, problem 14 (b) and (c)

(b) Must $R_1 \cup R_2$ be symmetric if R_1 and R_2 are?

Yes;
let $v,w,x,y \in S$
if R_1 is symmetric, $(v,w),(w,v) \in R_1$
if R_2 is symmetric, $(x,y),(y,x) \in R_2$
through the union, all symmetric pairs $(v,w),(w,v),(x,y),(y,x) \in (R_1 \cup R_2)$ and there is not the case where half of a symmetric pair is outside the union since both elements in the pair are guaranteed by the symmetry of one of the relations R_1 or R_2

(c) Must $R_1 \cup R_2$ be transitive if R_1 and R_2 are?

No;
by counterexample
let $v,w,x,y \in S$
if R_1 is transitive, $(v,w),(w,x),(v,x) \in R_1$
if R_2 is transitive, $(w,x),(x,y),(w,y) \in R_2$
so $(v,w),(w,x),(v,x),(w,x),(x,y),(w,y) \in (R_1 \cup R_2)$
but $(R_1 \cup R_2)$ is not transitive since, even though $(v,w),(w,y) \in (R_1 \cup R_2)$,
$(v,y) \notin (R_1 \cup R_2)$

9. Suppose S and T are two sets and f is a function from S to T. Let R_1 be an equivalence relation on T. Let R_2 be a binary relation on S such that $(x,y) \in R_2$ iff $(f(x_1,f(x_2)) \in R_1$. Prove that R_2 is also an equivalence relation.

A function from S to T is a special kind of relation R from S to T such that for each $x \in S$ there is exactly one $y \in T$ with $(x,y) \in R$ (this is the same as having a one-to-one and onto mapping). In addition, $f(x)$ is that unique element in T such that $(x,f(x))$ belongs to R_f.

Let $s \in S$ and $t \in T$. R_2 is defined only when $(f(x_1,f(x_2)) \in R_1$, meaning when $f(s,t) \in R_1$. So, for every $t \in T$ in R_1, there is a corresponding s that behaves the same under R_1. Because R_1 is an equivalence relation, every $s \in S$ must follow reflexive, symmetric and transitive laws. Since every $s \in S$ follows those laws, and R_2 is a relation on S, R_2 must also be an equivalence relation.

Also, like example 5 on pg. 168, you could just show that R_2 is reflexive, symmetric and transitive.