CS 1155 HW 5 Sample Solutions

Question 1. **(10points) Exercises 3.5, problem 5(b), page 174 of the text**

If G and H are both graphs with vertex set {1, 2, …, n}, we say that G is *isomorphic* to H, and write G H, in case there is a way to label the vertices of G so that it becomes H. For example, for the graphs in Figure 3, with vertex set {1, 2, 3}, are isomorphic by relabeling f(1) = 2, f(2) = 3 and f(3) = 1.

(b) Show that is an equivalence relation on the set of all graphs with vertex set {1, 2, …, n}

Solution:

Let the set of all graphs with vertex set {1, 2, …, n} be S, then is a relation on S.

If G H where G ∈ S and H ∈ S, the function f is a one-to-one function mapping the vertex set of G to that of H.

A relation is equivalent if and only if the relation is reflexive, symmetric and transitive (P. 167).

Therefore, we need to show is reflexive, symmetric and transitive.

(1). Show G G for every G ∈ S.

If G ∈ S, let V be the vertex set of G and the function f be such that

for every i ∈ V, f(i) = i

i.e. G G

∴ reflexive.

(2). Show if G H where G ∈ S and H ∈ S, then H G.

Let V_G be the vertex set of G and V_H that of H.

G H

∴ There is a function f such that

for every i ∈ V_G, f(i) = j where j ∈ V_H.

Since f is a one-to-one function, it is reversible, i.e. there is a function g such that

for every j ∈ V_H, g(j) = f^{-1}(j) = i where i ∈ V_G.

Labeling the vertexes of H using function g we get G.

∴ H G

∴ symmetric.

(3). Show if G H, H L where G, H, L ∈ S, then G L.

Let V_G be the vertex set of G, V_H that of H and V_L that of L.

Let function f and g be such that

for every $i \in V_G$, $f(i) = j$ where $j \in V_H$,
for every $j \in V_H$, $g(j) = k$ where $k \in V_L$

Let function $h = g \circ f$, i.e.

for every $i \in V_G$, $h(i) = g(f(i)) = g(j) = k$ where $k \in V_L$.

Labeling the vertexes of G using function h we get L.

$\therefore$ G L

$\therefore$ is transitive.

$\therefore$ is an equivalence relation.

Question 2. **(10 points) Exercises 3.5, problem 18, page 176 of the text.**

Let S be the set of all sequences (s_n) of real numbers, and define $(s_n) \approx (t_n)$ if $\{n \in \ : s_n \neq t_n\}$ is finite. Show that $\approx$ is an equivalence relation on S.

Solution:

Similar to Question 1, we need to show that $\approx$ is reflexive, symmetric and transitive.

(1). Show $(s_n) \approx (s_n)$ for every $s_n \in S$

Let set $T = \{n \in \ : s_n \neq s_n$ where $s_n \in S\}$. So number of elements in T is zero, hence finite.
$\therefore (s_n) \approx (s_n)$
$\therefore \approx$ is reflexive.

(2). Show if $(s_n) \approx (t_n)$ for every $s_n, t_n \in S$, then $(t_n) \approx (s_n)$

Let set $T = \{n \in \ : s_n \neq t_n$ where $s_n, t_n \in S\}$, set $T_1 = \{n \in \ : t_n \neq s_n$ where $s_n, t_n \in S\}$.
$(s_n) \approx (t_n)$
$\therefore$ T is finite.
T is equivalent to T_1.
$\therefore T_1$ is finite too.
$\therefore (t_n) \approx (s_n)$
$\therefore \approx$ is symmetric

(3). Show if $(s_n) \approx (t_n)$ and $(t_n) \approx (r_n)$ for every $s_n, t_n, r_n \in S$, then $(s_n) \approx (r_n)$

Let set $T = \{n \in \ : s_n \neq t_n$ where $s_n, t_n \in S\}$, set $T_1 = \{n \in \ : t_n \neq r_n$ where $t_n, r_n \in S\}$, set $T_2 = \{n \in \ : s_n \neq r_n$ where $s_n, r_n \in S\}$.
$(s_n) \approx (t_n)$ and $(t_n) \approx (r_n)$
$\therefore$ T and T_1 are finite.

Let $M_1 = \{s_n: n \in$ and is the same as in (s_n), and $s_n = t_n$ where $s_n, t_n \in S\}$,
$M_2 = \{t_n: n \in$ and is the same as in (t_n), and $s_n = t_n$ where $s_n, t_n \in S\}$,
$N_1 = \{s_n: n \in$ and is the same as in (s_n), and $s_n \neq t_n$ where $s_n, t_n \in S\}$,

and $N_2 = \{t_n: n \in$ and is the same as in (t_n), and $s_n \neq t_n$ where $s_n, t_n \in S\}$.
Apparently, M_1 and M_2 are equivalent, and N_1 and N_2 are finite.
Since (t_n) and (r_n) have finite number of different elements, M_2 and (r_n) have finite number of different elements.
$M_1 = M_2$
$\therefore$ M_1 and (r_n) have finite number of different elements.
Since N_1 is finite, it has finite number of different elements from (r_n).
$(s_n) = M_1 \cup N_1$,
$\therefore (s_n)$ and (r_n) have finite number of different elements.
$\therefore (s_n) \approx (r_n)$.
$\therefore \approx$ is transitive.

$\therefore \approx$ is an equivalent relation.

Question 3. **(15 points)**

Let A be a set with 10 elements.
(a). How many different binary relations on A are there?
(b). How many of them are reflexive?
(c). How many of them are symmetric?

Solution:

(a). Since all the binary relations on A is a subset of $A \times A$.
$\therefore$The number of different binary relations on A = $2^{|A \times A|} = 2^{100}$

(b). If R is a reflexive binary relation on A, then R should at least include the following tuples:
$(a_1, a_1), (a_2, a_2), \ldots, (a_{10}, a_{10})$.
Other tuples in relation R is a subset of $A \times A - \{(a_1, a_1), (a_2, a_2), \ldots, (a_{10}, a_{10})\}$.
$\therefore$ The number of different reflexive binary relation on A = $2^{|A \times A| - 10} = 2^{90}$

(c). If R is a symmetric relation on A, then R would be such that
if $(a_i, a_j) \in R$ where, $a_i, a_j \in A$, then $(a_j, a_i) \in R$.
Let $B = \{[(a_i, a_j), (a_j, a_i)]: a_i, a_j \in A \text{ and } i \neq j\}$ ([a, b] represents an unordered pair, so [a, b] = [b, a]).
$\therefore$ $|B| = (|A \times A| - 10) / 2 = 45$.
R must include a subset of the pairs in B plus a subset of $\{(a_1, a_1), (a_2, a_2), \ldots, (a_{10}, a_{10})\}$,
The number of different symmetric binary relation on A = $2^{|B|} \times 2^{10} = 2^{45} \times 2^{10} = 2^{55}$.

Question 4. **(15 points)**

Let R be a transitive and reflexive relation on a set A. Let T be a relation on A such that (a, b) is in T if and only if both (a, b) and (b, a) is in R. Prove that T is an equivalence relation.

Solution:

We must prove that T is reflexive, symmetric and transitive.

(1). Show $(a, a) \in T$ for every $a \in A$.
R is reflexive on A.
$\therefore$ For every $a \in A$, $(a, a) \in R$, i.e. let $b = a$, then $(a, b) = (b, a) \in R$
$\therefore$ $(a, a) \in T$.
$\therefore$ T is reflexive.

(2). Show if $(a, b) \in T$ where $a, b \in A$, then $(b, a) \in T$.
If $(a, b) \in T$, then both (a, b) and $(b, a) \in R$, i.e. both (b, a) and $(a, b) \in R$.
$\therefore$ $(b, a) \in T$.
$\therefore$ T is symmetric

(3). Show if $(a, b) \in T$ and $(b, c) \in T$, then $(a, c) \in T$.
If $(a, b) \in T$, then both (a, b) and $(b, a) \in R$.
Similarly both (b, c) and $(c, b) \in R$.
R is transitive
$\therefore$ $(a, c) \in R$ because (a, b) and $(b, c) \in R$.
$(c, a) \in R$ because (c, b) and $(b, a) \in R$.
$\therefore$ $(a, c) \in T$.
$\therefore$ T is transitive.

$\therefore$ T is an equivalence relation.

Question 5. **(10 points) Exercises 3.6, problem 8 (a), (b), page 183 of the text.**

(a). List the elements in the sets A_0, A_1 and A_2 defined by

$A_k = \{m \in \quad : -10 \le m \le 10$ and $m \quad k \pmod 3\}$.

(b). What is A_3? A_4? A_{73}?

Solution:

(a). $A_0 = \{m \in \quad : -10 \le m \le 10$ and $m \quad 0 \pmod 3\}$
$\therefore$ $A_0 = \{-9, -6, -3, 0, 3, 6, 9\}$
$A_1 = \{m \in \quad : -10 \le m \le 10$ and $m \quad 1 \pmod 3\}$
$\therefore$ $A_1 = \{-8, -5, -2, 1, 4, 7, 10\}$
$A_2 = \{m \in \quad : -10 \le m \le 10$ and $m \quad 2 \pmod 3\}$
$\therefore$ $A_2 = \{-10, -7, -4, -1, 2, 5, 8\}$

(b). $A_3 = \{m \in \quad : -10 \le m \le 10$ and $m \quad 3 \pmod 3\}$
$3 \quad 0 \pmod 3$
$\therefore$ $A_3 = A_0$
$A_4 = \{m \in \quad : -10 \le m \le 10$ and $m \quad 4 \pmod 3\}$
$4 \quad 1 \pmod 3$

$\therefore A_4 = A_1$

$A_{73} = \{m \in \quad : -10 \le m \le 10$ and $m \quad 73 \pmod 3\}$
$73 \quad 1 \pmod 3$
$\therefore A_{73} = A_1$

Question 6. **(15 points) Exercises 3.6, problem 14 (b), (c), and (d), page 184 of the text.**

(b). Is the function f: $[\quad]_p \rightarrow [\quad]_p$ given by $f([n]_p) = [n^2]_p$ well defined? Explain.
(c). Repeat part (b) for the function g: $[\quad]_6 \rightarrow [\quad]_{12}$ given by $g([n]_6) = [n^2]_{12}$.
(d). Repeat part (b) for the function h: $[\quad]_6 \rightarrow [\quad]_{12}$ given by $h([n]_6) = [n^3]_{12}$.

Solution:

(b). Let n + mp be the representative of $[n]_p$ where $m \in \quad$, then

$f([n]_p) = (n + mp)^2 \bmod p$
$= (n^2 + 2mp + m^2p^2) \bmod p$
$= (n^2 \bmod p + 2nmp \bmod p + m^2p^2 \bmod p) \bmod p$ (Corollary (a) on page 180)
$= (x + 0 + 0) \bmod p$
(Corollary (b) on page 180 and let $x = n^2 \bmod p$ where $x < p$)
$= x,$

which is independent of m.
$\therefore f([n]_p)$ doesn't depends on the representative of the equivalence class.
$\therefore f([n]_p)$ is well defined.

(c). Let n + 6m be the representative of $[n]_6$ where $m \in \quad$, then

$g([n]_6) = (n + 6m)^2 \bmod 12$
$= (n^2 + 12nm + 36m^2) \bmod 12$
$= (n^2 \bmod 12 + 12nm \bmod 12 + 36m^2 \bmod 12) \bmod 12$
$= (x + 0 + 0) \bmod 12$ (Let $x = n^2 \bmod 12$ where $x < 12$)
$= x,$

which is independent of m
$\therefore g([n]_6)$ doesn't depends on the representative of the equivalence class.
$\therefore g([n]_6)$ is well defined.

(d). Let n + 6m be the representative of $[n]_6$ where $m \in \quad$, then

$h([n]_6) = (n + 6m)^3 \bmod 12$
$= (n^3 + 18n^2m + 108nm^2 + m^3) \bmod 12$
$= (n^3 \bmod 12 + 18n^2m \bmod 12 + 108nm^2 \bmod 12 + 216m^3 \bmod 12) \bmod 12$
$= (y + 18xm \bmod 12 + 0 + 0) \bmod 12$
(Let $x = n^2 \bmod 12$ and $y = n^3 \bmod 12$ where $x, y < 12$)

= y + 18xm mod 12

which is dependent of m

$\therefore$ h($[n]_6$) depends on the representative of the equivalence class.

$\therefore$ h($[n]_6$) is not well defined.

Question 7. **(10 points) Exercises 3.6, problem 16 (a), (b), page 184 of the text.**

(a). Show that the four-digit number n = abcd is divisible by 2 if and only if the last digit d is.

(b). Show that n = abcd is divisible by 5 if and only if d is.

Solution:

(a). n = abcd = 1000a + 100b + 10c + d

$\therefore$ n mod 2 = (1000a + 100b + 10c + d) mod 2

= 1000a mod 2 + 100b mod 2 + 10c mod 2 + d mod 2 (Corollary (a) on page 180)

= 0 + 0 + 0 + d mod 2 (Corollary (b) on page 180)

= d mod 2

$\therefore$ n mod 2 = 0 if and only if d mod 2 = 0.

$\therefore$ n is divisible by 2 if and only if the last digit d is.

(b). n = abcd = 1000a + 100b + 10c + d

$\therefore$ n mod 5 = (1000a + 100b + 10c + d) mod 5

= 1000a mod 5 + 100b mod 5 + 10c mod 5 + d mod 5

= 0 + 0 + 0 + d mod 5

= d mod 5

$\therefore$ n mod 5 = 0 if and only if d mod 5 = 0.

$\therefore$ n is divisible by 5 if and only if the last digit d is.

Question 8. **(15 points) Exercises 4.2, problem 16, page 211 of the text.**

Prove (2n + 1) + (2n + 3) + (2n + 5) + … + (4n – 1) = $3n^2$ for all n in . The sum can also be written $\sum_{i=n}^{2n-1}(2i+1)$**.**

Solution:

Prove by induction.

(1). When n = 1,

$$\sum_{i=n}^{2n-1}(2i+1) = \sum_{i=1}^{1}(2i+1) = 2 \times 1 + 1 = 3$$

$3n^2 = 3 \times 1^2 = 3$

$\therefore \sum_{i=n}^{2n-1}(2i+1)$ = $3n^2$

The claim is true when n = 1.

(2). Suppose that the claim is true when n = k. We need to show that the claim is also true when n = k + 1.

$\therefore \sum_{i=k}^{2k-1}(2i+1)$ = (2k + 1) + (2k + 3) + (2k + 5) + … + (4k – 1) = $3k^2$

$\sum_{i=k+1}^{2(k+1)-1}(2i+1)$ = [2(k + 1) + 1] + [2(k + 2) + 1] + …

+ [2(2(k + 1) – 3) + 1] + [2(2(k + 1) – 2) + 1] + [2(2(k + 1) – 1) + 1]

= (2k + 3) + (2k + 5) + … + (4k – 1) + (4k + 1) + (4k + 3)

= [(2k + 1) + (2k + 3) + (2k + 5) + … + (4k – 1)]

– (2k + 1) + (4k + 1) + (4k + 3)

= $3k^2 + 6k + 3$

= $3(k^2 + 2k + 1)$

= $3(k + 1)^2$

$\therefore$ If the claim is true when n = k, then it is also true when n = k + 1.

From (1) and (2), $\sum_{i=n}^{2n-1}(2i+1)$ = $3n^2$ is true.