CS 1155 HW6 Sample Solutions

1. (10 points) Exercises 4.2, problem 18, page 211 of the text.

Prove $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots n)^2$, i.e., $\sum_{i=1}^{n} i^3 = [\sum_{i=1}^{n} i]^2$ for all n in $\mathbb{P}$. Hint: Use the identity in Example 2(b).

Basis:

n = 1: $\sum_{i=1}^{1} i^3 = [\sum_{i=1}^{1} i]^2$

$1^3 = 1^2$
1 = 1
true

Assume inductively that the claim is true for some positive integer n = k, i.e., that $\sum_{i=1}^{k} i^3 = [\sum_{i=1}^{k} i]^2$. Need to show that this assumption implies that the claim is true for n = k + 1.

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k + 1)^3$$

$$= [\sum_{i=1}^{k} i]^2 + (k + 1)^3$$

$$= (k(k+1)/2)^2 + (k + 1)^3$$
$$= (k + 1)^2(k/4 + (k + 1))$$
$$= (k + 1)^2((k + 4k + 4)/4)$$
$$= (k + 1)^2((k + 2)/2)^2$$
$$= (k + 1)^2[((k + 1) +1)/2]^2$$
$$= [(k + 1)((k + 1) + 1)/2]^2$$

$$= [\sum_{i=1}^{k+1} i]^2$$

2. **(10 points) Exercises 4.2, problem 22, page 211 of the text.**
Prove that $8^{n+2} + 9^{2n+1}$ is divisible by 73 for $n \in \mathbb{N}$.

Basis:
n = 0: $8^{0+2} + 9^{2(0)+1}$
$= 8^2 + 9$
$= 64 + 9$
$= 73$
which is divisible by 73

Assume inductively that the claim is true for some n = k. Show that the assumption implies that the claim is true for n = k + 1.

$8^{(k+1)+2} + 9^{2(k+1)+1} = 8^{k+3} + 9^{2k+3}$
$= 8(8^{k+2} + 9^{2k+1}) + 9^{2k+3} - 8(9^{2k+1})$

By the induction hypothesis $8(8^{k+2} + 9^{2k+1})$ is divisible by 73.
So must show that $9^{2k+3} - 8(9^{2k+1})$ is divisible by 73.

$9^{2k+3} - 8(9^{2k+1}) = 9^{2k+1}(9^2 - 8)$
$= 9^{2k+1}(81 - 8)$
$= 9^{2k+1}(73)$
which is divisible by 73

3. **(15 points) Exercises 4.4, problem 8, page 23 of the text.**
Let $\Sigma = \{a, b\}$ and let s_n denote the number of words of length n that do not contain the string *ab*.

(a) Calculate s_0, s_1, s_2 and s_3.

$A_0 = \{\lambda\}$
$A_1 = \{a, b\}$
$A_2 = \{aa, bb, ba\}$
$A_3 = \{aaa, bbb, baa, bba\}$

so
$s_0 = 1$
$s_1 = 2$
$s_2 = 3$
$s_3 = 4$

(b) Find a formula for s_n and prove it is correct.

This is similar to Example 4(a) on page 228.
To get a recurrence formula for s_n, consider $n \geq 1$. If a word in A_n ends in *a*, then *a* can be preceded by any word in A_{n-1}. So s_{n-1} words in A_n end in *a*. If a word A_n ends in *b*, then it must be all *b*'s. So 1 word in A_n ends in *b*. Thus, $s_n = s_{n-1} + 1$ is the recurrence relation for s_n and

$s_n = s_{n-1} + 1 = s_{n-2} + 1 + 1 = s_{n-3} + 1 + 1 + 1 = \ldots$

$\quad = s_{n-1} + 1 = s_{n-2} + 2(1) = s_{n-3} + 3(1) = \ldots = s_{n-n} + n(1) = s_0 + n = 1 + n$

To show that this formula is correct, need to prove that it satisfies the recurrence conditions that define s_n, namely that $s_1 = 2$ and $s_n = s_{n-1} + 1$

$s_n = 1 + n$
$s_1 = 1 + 1$
$2 = 2$
true

$s_n = s_{n-1} + 1$
$1 + n = (1 + n - 1) + 1$
$1 + n = 1 + n$
true

4. **(15 points) Exercises 4.4, problem 12, page 234 of the text. Prove that the sequence $M_1, M_2, \ldots$ of matrices defined by**

$$\mathbf{M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} \quad \textbf{and}$$

$$\mathbf{M_n = \begin{bmatrix} FIB(n + 1) & FIB(n) \\ FIB(n) & FIB(n - 1) \end{bmatrix}}$$

for $n \geq 3$ satisfies $M_n + 1 = M_1 \bullet M_n$ for $n \in$ P.

Basis:
$n = 3$: $M_{3+1} = M_1 \bullet M_3$
$\quad M_4 = M_1 \bullet M_3$

$$\begin{bmatrix} FIB(4 + 1) & FIB(4) \\ FIB(4) & FIB(4 - 1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} FIB(3 + 1) & FIB(3) \\ FIB(3) & FIB(3 - 1) \end{bmatrix}$$

$$\begin{bmatrix} FIB(5) & FIB(4) \\ FIB(4) & FIB(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} FIB(4) & FIB(3) \\ FIB(3) & FIB(2) \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1(3) + 1(2) & 1(2) + 1(1) \\ 1(3) + 0(2) & 1(2) + 0(1) \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

true

Assume inductively that the claim is true for n = k. Then, show that the assumption implies the claim is true for n = k + 1.

$M_{k+1+1} = M_1 \bullet M_{k+1}$

substitute using the inductive hypothesis

$M_{k+2} = M_1 \bullet M_1 \bullet M_k$

$M_{k+2} = M_1 \bullet M_1 \bullet M_1 \bullet M_{k-1}$

…

$M_{k+2} = M_1 \bullet M_1 \bullet M_1 \bullet \ldots \bullet M_1$

there are k + 2 M_1's

$M_{k+2} = M^{k+2}$

substitute using the definition of the sequence and Example 7, page 231

$$\begin{bmatrix} FIB(k+2+1) & FIB(k+2) \\ FIB(k+2) & FIB(k+2-1) \end{bmatrix} = \begin{bmatrix} FIB(k+2+1) & FIB(k+2) \\ FIB(k+2) & FIB(k+2-1) \end{bmatrix}$$

true

5. **(10 points)**

Consider the definition of a Fibonacci sequence FIB(n) in example 3(a), page 228 of the text. Prove, by induction, that for all integers k ≥ 1,

$$\mathbf{FIB_{k+2} = 1 + \sum_{i=1}^{k} FIB_i.}$$

Basis:

k = 1: $FIB_{1+2} = 1 + \sum_{i=1}^{1} FIB_i$

$FIB_3 = 1 + 1$

$2 = 2$

true

Assume inductively that the claim is true for some positive k. Need to show that the assumption implies the claim is true for k + 1.

$$FIB_{(k+1)+2} = 1 + \sum_{i=1}^{k+1} FIB_i$$

$$FIB_{k+3} = 1 + FIB_{k+1} + \sum_{i=1}^{k} FIB_i$$

substitute the inductive hypothesis

$FIB_{k+3} = FIB_{k+1} + FIB_{k+2}$

true from the definition of the Fibonacci sequence

FIB(n) = FIB(n - 1) + FIB(n - 2) where k + 3 is substituted for n.

6. **(10 points) Let m = α + β and a = αβ, where m ≠ 1 and α ≠ β.**
Let A_2 = m - a/(m - 1). For k > 1, let A_{k+1} = m - a/A_k. Prove that, for n ≥ 2,

$$\mathbf{A_n = [(\alpha^{n+1} - \beta^{n+1}) - (\alpha^n - \beta^n)]/[(\alpha^n - \beta^n) - (\alpha^{n-1} - \beta^{n-1})]}$$

Basis:

n = 2: $A_2 = [(\alpha^{2+1} - \beta^{2+1}) – (\alpha^2 - \beta^2)]/[(\alpha^2 - \beta^2) – (\alpha^{2-1} - \beta^{2-1})]$

$A_2 = [(\alpha^3 - \beta^3) – (\alpha^2 - \beta^2)]/[(\alpha^2 - \beta^2) – (\alpha - \beta)]$

$A_2 = [(\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2) – (\alpha - \beta)(\alpha + \beta)]/[(\alpha - \beta)(\alpha + \beta) - (\alpha - \beta)]$

$A_2 = [(\alpha - \beta)((\alpha^2 + \alpha\beta + \beta^2) – (\alpha + \beta))]/[(\alpha - \beta)(\alpha + \beta - 1)]$

$A_2 = [(\alpha^2 + \alpha\beta + \beta^2) – (\alpha + \beta)]/(\alpha + \beta - 1)$

$A_2 = [(\alpha + \beta)^2 - 2\alpha\beta + \alpha\beta – (\alpha + \beta)]/(\alpha + \beta - 1)$

$A_2 = (m^2 – 2a + a – m)/(m – 1)$
$A_2 = [m(m – 1) –a]/(m – 1)$
$A_2 = m – a/(m – 1)$
true

Assume inductively that the claim is true for some positive k. Need to show that the assumption implies the claim is true for k + 1.

$A_{k+1} = m – a/A_k$
$A_{k+1} = \alpha + \beta - \alpha\beta/\{[(\alpha^{k+1} - \beta^{k+1}) – (\alpha^k - \beta^k)]/[(\alpha^k - \beta^k) – (\alpha^{k-1} - \beta^{k-1})]\}$
$A_{k+1} = \alpha + \beta - \alpha\beta[(\alpha^k - \beta^k) – (\alpha^{k-1} - \beta^{k-1})]/ [(\alpha^{k+1} - \beta^{k+1}) – (\alpha^k - \beta^k)]$
$A_{k+1} = \{(\alpha + \beta)[(\alpha^{k+1} - \beta^{k+1}) – (\alpha^k - \beta^k)] - \alpha\beta[(\alpha^k - \beta^k) – (\alpha^{k-1} - \beta^{k-1})]/ [(\alpha^{k+1} - \beta^{k+1}) – (\alpha^k - \beta^k)]$
$A_{k+1} = [(\alpha + \beta)(\alpha^{k+1} - \beta^{k+1} – \alpha^k + \beta^k) - \alpha\beta(\alpha^k - \beta^k – \alpha^{k-1} + \beta^{k-1})/ [(\alpha^{k+1} – \beta^{k+1}) – (\alpha^k - \beta^k)]$
$A_{k+1} = (\alpha^{k+2} - \alpha\beta^{k+1} - \alpha^{k+1} + \alpha\beta^k + \alpha^{k+1}\beta - \beta^{k+2} - \alpha^k\beta + \beta^{k+1} - \alpha^{k+1}\beta + \alpha\beta^{k+1} + \alpha^k\beta - \alpha\beta^k)/ [(\alpha^{k+1} – \beta^{k+1}) – (\alpha^k - \beta^k)]$
$A_{k+1} = (\alpha^{k+2} - \alpha^{k+1} - \beta^{k+2} + \beta^{k+1})/ [(\alpha^{k+1} – \beta^{k+1}) – (\alpha^k - \beta^k)]$
$A_{k+1} = [(\alpha^{(k+1)+1} - \beta^{(k+1)+1}) – (\alpha^{k+1} - \beta^{k+1})]/ [(\alpha^{k+1} – \beta^{k+1}) – (\alpha^{(k+1)-1} – \beta^{(k+1)-1})]$
true

7. (15 points) Let R be a binary relation on a set S with n elements. Let $R_0 = R$. Define R_{i+1} from R_i as follows:

$$R_{i+1} = R_i \cup \{(a, c) \mid (a, b) \in R_i \text{ and } (b, c) \in R_i\}.$$

Show that if $R_i = R_{i+1}$ for some $i \geq 0$, then $R_i = R_j$ for all $j > i$.

Basis:
i = 0: $R_{0+1} = R_0 \cup \{(a, c) \mid (a, b) \in R_0 \text{ and } (b, c) \in R_0\}$
$R_1 = R_0 \cup \{\text{subset of } R_0\}$
$R_1 = R_0$
true

Assume inductively that the claim is true for all positive integers $0 \leq i < k$. Need to show that the claim is true for k + 1.

If we set $k = i + 1$, then the assumption holds since $0 \leq i < k$ and $R_i = R_{i+1}$
So $k + 1 = i + 2$
$R_{i+2} = R_{i+1} \cup \{\text{subset of } R_{i+1}\}$
$R_{i+2} = R_{i+1}$
Substitute using the induction hypothesis
$R_{i+2} = R_i$

So for $k + 1 = i + 2 = j$, where $0 \leq i < j$ the assumption still holds and $R_j = R_i$

8. Consider the recursive definition of the language L as follows: L consists of all strings over {0, 1} obtained from the basis step by a finite number of applications on the recursive step:
Basis: The empty string is in L;
Recursive: If $x \in L$ then $11x0 \in L$.

(a) (5 points) How many strings of length k exist for each $k \geq 0$?

for strings of length 3k, there is 1 string
for strings any other length, there are 0 strings

$s_n = 1$ if n is divisible by 3
$\quad = 0$ otherwise

(b) (10 points) Prove, by induction, that for every string $w \in L$ the number of 1's is twice the number of 0's.

Note that each string increases in length by 3 (add two 1's and one 0) to make the next string. So one can write a recurrence relation based on the length k.
$k_n = 3 + k_{n-1}$
which is the same as
$k_n = |11| + |0| + k_{n-1}$
where k_n is the length of the nth string in the language and you add the length of two 1's and the length of one 0 since these are always going to be put onto the previous string to make the next one.

Basis:
n = 1: $k_1 = 2 + 1\ k_0$ (k_0 is λ, and k_1 is 110)
$3 = 3 + 0$
$3 = 3$
true

Assume inductively that the claim is true for all positive integers $0 \leq n$. Need to show that the claim is true for n + 1.

$k_{n+1} = |11| + |0| + k_n$
$k_{n+1} = |11| + |0| + |11| + |0| + k_{n-1}$
From the inductive hypothesis, we know that k_{n-1} has twice as many 1's as there are 0's. From inspection, we see that $|11| + |0| + |11| + |0|$ has twice as many 1's (a total of 4) as there are 0's (a total of 2). So we see that for every 0 added to a string there are two 1's added. Thus for every string $w \in L$, the number of 1's is twice the number of 0's.