CS 1155 Spring 1999 HW#7 Sample Solutions

1. Consider the following recurrence equation: $s_n = 6s_{n-1} - 11s_{n-2} + 6s_{n-3}$, with $s_0 = 3$, $s_1 = 6$, and $s_2 = 14$.

(a) (5 points) What is the characteristic equation of this recurrence?

Solution:

It is reasonable to hope that s^n would have the form $s_n = cr^n$ for a nonzero constant c. This would mean that

$$cr^n = 6cr^{n-1} - 11cr^{n-2} + 6cr^{n-3}$$

Dividing by cr^{n-3} would then give

$$r^3 = 6r^2 - 11r + 6$$

Therefore, the characteristic equation of the recurrence relation would be

$$x^3 - 6x^2 + 11x - 6 = 0$$ ∎

(b) (5 points) What are the roots of the characteristic equation?

Solution:

The characteristic equation is equivalent to the following:

$$r^3 - 6r^2 + 9r + 2r - 6 = 0$$
$$(r - 3)^2 r + 2(r - 3) = 0$$
$$(r - 3)[(r - 3)r + 2] = 0$$
$$(r - 3)(r - 2)(r - 1) = 0$$

Clearly the roots of the characteristic equation are 1, 2, and 3. ∎

(c) (10 points) What is the solution of the recurrence? Prove your answer.

Solution:

Since the characteristic equation has three different roots, we can assume that

$$s_n = c_1 r_1^{\,n} + c_2 r_2^{\,n} + c_3 r_3^{\,n}$$

We know that $s_0 = 3$, $s_1 = 6$, and $s_2 = 14$. Let $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$. Solving

$$\begin{cases} 3 = c_1 + c_2 + c_3 \\ 6 = c_1 + 2c_2 + 3c_3 \\ 14 = c_1 + 4c_2 + 9c_3 \end{cases}$$

gives us $c_1 = c_2 = c_3 = 1$. In other words, the solution of the recurrence is

$$s_n = r_1^{\,n} + r_2^{\,n} + r_3^{\,n} = 1 + 2^n + 3^n$$

We can prove this by using induction. The basis cases where $n = 0, 1, 2$, can be verified easily. Assuming that for any n so that $0 \le n \le m$, our solution is correct, we want to prove that for $n = m + 1$, our solution is also correct:

$$s_n = s_{m+1}$$
$$= 6s_m - 11s_{m-1} + 6s_{m-2}$$
$$= 6(1 + 2^m + 3^m) - 11(1 + 2^{m-1} + 3^{m-1}) + 6(1 + 2^{m-2} + 3^{m-2})$$
$$= (6 - 11 + 6) + (6 \times 2^2 - 11 \times 2 + 6) \times 2^{m-2} + (6 \times 3^2 - 11 \times 3 + 6) \times 3^{m-2}$$

$= 1 + 2^{m+1} + 3^{m+1}$

Therefore, our solution of the recurrence is correct. ∎

2. (10 points) Consider the sequence defined in Example 4, page 239 of the text. The characteristic equation has only one root, namely $r = 3$. (Therefore, for the solution of this recurrence, part (b) of Theorem 1 is applicable.) The proof of part (a) of Theorem 1 has two steps: (a) a basis step where the constants c_1, c_2 are determined, and (b) and inductive step. Which of these two steps in the proof of part (a) of Theorem 1 fails for this example? Why?

   Solution:

   The first step fails because the equations

$$\begin{cases} s_0 = c_1 + c_2 \\ s_1 = c_1 r_1 + c_2 r_2 \end{cases}$$

   cannot be solved for c_1 or c_2 if the characteristic equation has only one root, i.e. $r_1 = r_2$. ∎

3. (15 points) Consider the matrices M^1, M^2, and M^n defined on page 231 of the text book. Using these definitions, prove that, for all $n \geq 1$, $FIB(n+1) \cdot FIB(n-1) - FIB^2(n) = (-1)^n$.

   Solution:

   We could use induction for this problem, but a simpler route exists:

$$FIB(n+1) \cdot FIB(n-1) - FIB^2(n)$$

$$= \left| \begin{bmatrix} FIB(n+1) & FIB(n) \\ FIB(n) & FIB(n-1) \end{bmatrix} \right| = \left| M^n \right| = \left| M^1 \right|^n = \left| \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right|^n = (-1)^n$$ ∎

4. (10 points) Consider the sequence s_n where $s_0 = 2$, $s_1 = 1$ and $s_n = s_{n-1} + s_{n-2}$ for $n \geq 2$.

   (a) Calculate s_n for $n = 2, 3, 4, 5$ and 6.

   Solution:

   $s_2 = s_1 + s_0 = 1 + 2 = 3$
   $s_3 = s_2 + s_1 = 3 + 1 = 4$
   $s_4 = s_3 + s_2 = 4 + 3 = 7$
   $s_5 = s_4 + s_3 = 7 + 4 = 11$
   $s_6 = s_5 + s_4 = 11 + 7 = 18$ ∎

   (b) Give an explicit formula for s_n.

   Solution:

   Theorem 1 is applicable in this case. The characteristic equation is

$$x^2 - x - 1 = 0$$

   It has two roots: $\frac{1 + \sqrt{5}}{2}$ and $\frac{1 - \sqrt{5}}{2}$. Solving

$$\begin{cases} 2 = c_1 + c_2 \\ 1 = \frac{1+\sqrt{5}}{2}c_1 + \frac{1-\sqrt{5}}{2}c_2 \end{cases}$$

gives us $c_1 = c_2 = 1$. Therefore,

$$s_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$ ∎

5. (10 points) Give an explicit formula for s_n.

(c) $s_0 = 5$ and $s_n = s_{n-1}$ for $n \geq 1$.

Solution:

$s_n = 5$ ∎

(d) $s_0 = 3$, $s_1 = 5$ and $s_n = 2s_{n-1} - s_{n-2}$ for $n \geq 2$.

Solution:

Using Theorem 1, the characteristic equation is

$$x^2 - 2x + 1 = 0$$

which has only one root, 1. Solving

$$\begin{cases} 3 = c_1 + c_2 \cdot 0 \\ 5 = c_1 + c_2 \cdot 1 \end{cases}$$

we have $c_1 = 3$ and $c_2 = 2$. Therefore

$$s_n = 3 + 2n$$ ∎

6. (10 points) Theorem 2 does not specifically apply to the following recurrence, but the ideas carry over.

(b) Given that $t_{3n} = 3t_n + f(n)$, find a formula for t_{3^m}.

Solution:

Our guess would be

$$s_{3^m} = 3^m \cdot \left(s_1 + \frac{1}{3}\sum_{i}^{m-1} \frac{f(3^i)}{3^i}\right) \qquad (\S)$$

and we can prove it by using induction. The basis is

$$s_1 = s_{3^0} = 3^0 \cdot \left(s_1 + \frac{1}{3} \times 0\right) = s_1$$

Assume that (§) is true for a specific m,

$$
\begin{aligned}
s_{3^{m+1}} &= s_{3 \cdot 3^m} \\
&= 3 \cdot s_{3^m} + f(3^m) \\
&= 3^{m+1} \cdot \left(s_1 + \frac{1}{3} \sum_i^{m-1} \frac{f(3^i)}{3^i} \right) + f(3^m) \\
&= 3^{m+1} \cdot \left(s_1 + \frac{1}{3} \sum_i^{m-1} \frac{f(3^i)}{3^i} + \frac{f(3^m)}{3 \cdot 3^m} \right) \\
&= 3^{m+1} \cdot \left(s_1 + \frac{1}{3} \sum_i^{m} \frac{f(3^i)}{3^i} \right)
\end{aligned}
$$

The inductive step is also true. Therefore (§) is true for all $m \geq 0$. ∎

7. (10 points) Recursively define $a_0 = 1$, $a_1 = 3$, $a_2 = 5$ and $a_n = 3a_{n-2} + 2a_{n-3}$ for $n \geq 3$.

(d) Prove that $a_n = 2a_{n-1} + (-1)^{n-1}$ for $n \geq 1$. (§)

Solution:

We need to use the *Second Principle of Mathematical Induction* for this problem. First, we need to verify that (§) is true for $n = 1$ and 2.

if $n = 1$, $2a_{n-1} + (-1)^{n-1} = 2a_{1-1} + (-1)^{1-1} = 2a_0 + 1 = 3 = a_1$

if $n = 2$, $2a_{n-1} + (-1)^{n-1} = 2a_{2-1} + (-1)^{2-1} = 2a_1 - 1 = 5 = a_2$

Then, assuming that (§) is true for indices from 1 to $n - 1$, we need to prove that (§) is still true for index n.

$$
\begin{aligned}
a_n &= 3a_{n-2} + 2a_{n-3} \\
&= 3(2a_{n-3} + (-1)^{n-3}) + 2(2a_{n-4} + (-1)^{n-4}) \\
&= 6a_{n-3} + 3(-1)^{n-3} + 4a_{n-4} + 2(-1)^{n-4} \\
&= 6a_{n-3} + 4a_{n-4} + (-3 + 2)(-1)^{n-4} \\
&= 6a_{n-3} + 4a_{n-4} + (-1)^{n-3} \\
&= 2(3a_{n-3} + 2a_{n-4}) + (-1)^2(-1)^{n-3} \\
&= 2a_{n-1} + (-1)^{n-1}
\end{aligned}
$$

The proof is not complete at this point. Notice that in order to use the (§) assumption in the second sub-step above, $n - 3$ has to be ≥ 1. This means that our inductive step works only if $n \geq 4$ and the $n = 3$ case has to be covered in the induction basis:

$a_3 = 3a_1 + 2a_0 = 9 + 2 = 11$

if $n = 3$, $2a_{n-1} + (-1)^{n-1} = 2a_{3-1} + (-1)^{3-1} = 2a_2 + 1 = 11 = a_3$

Now the proof is complete. ∎

8. Let p, q be two propositions. Consider the following recursive definition of the set E of well-formed conjunctive and disjunctive Boolean expressions using p and q:

   Basis: $p, q \in$ E.

   Recursive Step: If e_1, e_2 are expressions in E, then $(e_1 \vee e_2) \in E$ and $(e_1 \wedge e_2) \in E$.

   An expression is in E iff it is obtained from the basis by a finite number of applications of the recursive step.

   (a) (5 points) What are the expressions obtained after two applications of the recursive step to the set of expressions defined by the basis?

   Solution:

   Let E_i denote the set E after i applications of the recursive step.

i	E_i
0	p, q
1	p, q, $(p \wedge p), (p \vee p), (q \wedge q), (q \vee q), (p \wedge q), (p \vee q), (q \wedge p), (q \vee p)$
2	p, q, $(p \wedge p), (p \vee p), (q \wedge q), (q \vee q), (p \wedge q), (p \vee q), (q \wedge p), (q \vee p)$, plus all $10 \times 10 \times 2 = 200$ cases of $(e_1 \wedge e_2)$ and $(e_1 \vee e_2)$, where $e_1, e_2 \in E_1$

   (b) (10 points) For an expression e, let $P(e)$ denote the number of occurrences of the propositions p and q in e and let $O(e)$ denote the number of occurrences of the operators $\wedge$ and $\vee$ in e. Prove, by induction, that for every expression $e \in E$, $P(e) = 1 + O(e)$.

   Solution:

   Induction basis: for every expression $e \in E_0 = \{p, q\}$, $P(e) = 1 + O(e)$, because

   $P(p) = 1 = 1 + O(p)$, $P(q) = 1 = 1 + O(q)$

   Inductive step: assuming that for every expression $e \in E_i$, $P(e) = 1 + O(e)$, where $0 \le i \le n$, we want to prove that for every expression $e \in E_{n+1}$,

   $$P(e) = 1 + O(e) \qquad (§)$$

   If $e \in E_n$, (§) is obviously true according to the inductive assumption. Therefore we only need to consider the cases where $e \in E_{n+1}$ but $e \notin E_n$. Then e can be either $(e_1 \wedge e_2)$ or $(e_1 \vee e_2)$, where $e_1 \in E_i$, $e_2 \in E_j$, $0 \le i \le n$, $0 \le j \le n$. If $e = (e_1 \wedge e_2)$,

   $$\begin{aligned} P(e) &= P((e_1 \wedge e_2)) = P(e_1) + P(e_2) \\ &= (1 + O(e_1)) + (1 + O(e_2)) \\ &= 1 + (1 + O(e_1) + O(e_2)) \\ &= 1 + O((e_1 \wedge e_2)) \\ &= 1 + O(e) \end{aligned}$$

   The $e = (e_1 \vee e_2)$ case is very similar.

   In summary, according to the *Second Principle of Mathematical Induction*, (§) is true for all E. ∎