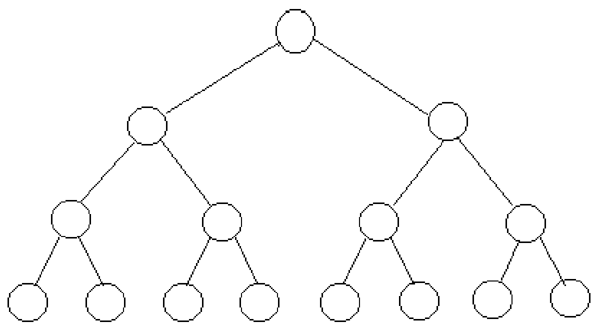
CS1155 HW 8 Solutions

1. **Consider the following recursive definition of a tree:**

   [See hw assignment]

   a) **(5 points) Draw the graph obtained by applying the recursive step three times starting from the basis.**

   The tree is a complete binary tree.



   b) **(10 points) Prove, by induction, that the number of vertices in the tree that results by applying the recursive step k times starting from the basis is $\mathbf{2^{k+1} – 1}$.**

   Basis: k=0

   The vertex set is {r}, so there is 1 vertex in the tree.
   $2^{0+1} – 1 = 2 – 1 = 1$.

   Induction Hypothesis: The number of vertices in the tree after k steps is $2^{k+1} – 1$.
   Induction Step:

   Prove the number of vertices in the tree after k+1 steps is $2^{k+2} – 1$.

   After the k+1[st] step, the vertex set of V = V' U {u(1), u(2), …. u(2n)}. V' is the vertex set of the tree after k steps. By the induction hypothesis, we know that $|V'| = 2^{k+1} – 1$.

   So we need to determine how many vertices are in {u(1).. u(2n)} = 2n. Now we need to determine what n is. This is the size of the leaf set of the tree after the previous iteration. This is the set of vertices that were added in the previous iteration. Therefore in each iteration we are adding twice the number of nodes we added in the previous iteration (from n to 2n). In the basis, we have 1 leaf vertex.

   Therefore in the first iteration step, we add 2 vertices. If we are doubling in each iteration step, then in the m-th step is 2^m. In the k+1[st] step, $n=2^k$. So $2n = 2^{k+1}$. So $|V| = |V'| + 2^{k+1}$

$= 2^{k+1} - 1 + 2^{k+1} = 2(2^{k+1}) - 1 = 2^{k+2} - 1.$

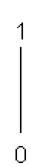
2. **(15 points) Let f [x(1)..x(n)] be the boolean function on B^n that takes the value 1 at [a(1)..a(n)] $\in B^n$ iff an even number of the entries a(1).. a(n) are 1. Show that any Boolean expression in the sum-of products literals form for f is a sum of at least 2^{n-1} products and each of these products must have n literals.**

   Step 1: Show that there must be 2^{n-1} products.
   Since each a(k) is an element of B, it can only be a 0 or a 1. Therefore there are 2 choices for each a(k), and therefore there are 2^n possible minterms. Half of these minterms will have an even number of 1’s (start with all 0’s and have to change 2 at a time – so we are alternating down the ordered of minterms). Therefore there are 2^{n-1} minterms necessary to express all the cases where an even number of entries are 1. Because they are minterms, they have n literals.

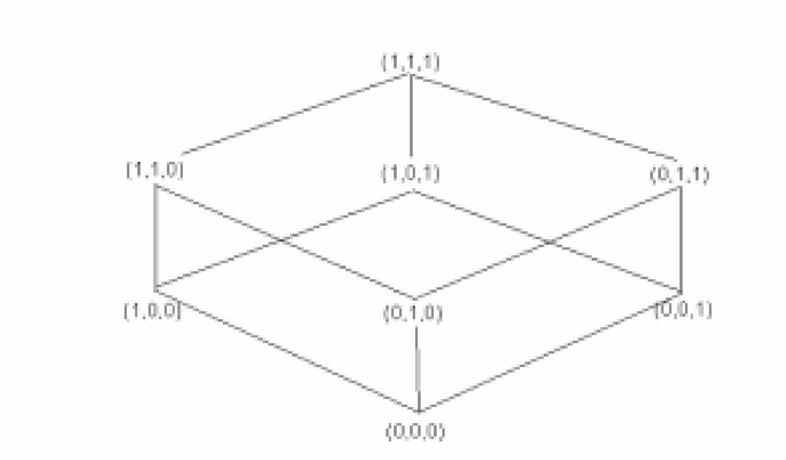
   Step 2: Prove that there can’t be a form with less products/literals.
   Proof by contradiction: Assume that there is a form with less than 2^{n-1} products. In that case one of these terms must have less than n literals (by step 1). There must be a k then, such that neither a(k) nor a(k)’ is in that product. We know that the product has an even number of 1’s in it, otherwise it would not satisfy f. But then if we choose a(k) as 1, we now have an odd number of 1’s in the series, and yet this series [a(1)..a(n)] still satisfies f. This is a contradiction, and therefore there can not be a form with less than 2^{n-1} products of n literals.

3. **(10 points) Ex. 9.1, problem 9 b, d.**
   Draw Hasse diagrams of the following Boolean algebras. Draw the element 1 at the top and direct the edges generally downwards.

   b. B



d. B^3



4. (10 points) Ex. 9.3, problem 4 a, b.
Consider the Boolean expression x v yz in x,y,z.

a) Give a table for the corresponding Boolean function f: B^3 -> B

x	y	z	yz	x v yz
1	1	1	1	1
1	1	0	0	1
1	0	1	0	1
1	0	0	0	1
0	1	1	1	1
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

b) Write the expression in minterm canonical form.

f = xyz + xyz’ + xy’z + xy’z’ + x’yz

5. (5 points) Ex. 9.3, problem 8 b
The Boolean function f: B^3 -> B is given by f(a,b,c) = a +2 b +2 c for (a,b,c) in B^3. Recall that +2 refers to addition modulo 2, which is defined in sec 3.6.

b. Write the expression in minterm canonical form with variables x, y, z.

F is the function that is true when an odd number of a, b, and c are true.
Therefore the expression is:
f = xyz + xy’z’ + x’yz’ + x’y’z

6. [ignored]

7. **(10 points) Ex 9.3, problem 11 b**
   There is anotion of maxterm dual to the notion of minterm. A maxterm in x(1)..x(n) is a join of n literals, each involving a different one of x(1)..x(n).

   b. Write xy' V x'y as a product of maxterms in x and y

   The trick is to realize that you can add joins that are always false to the desired expression without changing it.
   So rewrite the above expression as:
   xx' V xy' V x'y V y'y
   (xx' V xy') V (x'y V y'y)
   x(x' V y') V y(x' V y')
   (x V y)(x' V y')

8. **(10 points) Ex 10.3 problem 1b**
   For each of the following Boolean matrices, consider the corresponding relation R on {1, 2, 3}. Find the Boolean matrix for R^2 and determine whether R is transitive.

   R^2 = A*A

   $$= \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix} * \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix} = \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$$

   Since R^2 is a subset of R (R has a 1 everywhere R^2 does), R is transitive.

9. **(10 points) Consider the matrix A defined for the relation R in problem 5(a) page 577 of the text. Prove by induction that the matrix for R^n=A*A.**

   $$\mathbf{A} = \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\end{bmatrix}$$

   $$A*A = \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\end{bmatrix} * \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&1\\1&0&1\end{bmatrix}$$

   Basis: n=2
   R^2 = A*A by definition.

   Induction Hypothesis: R^n = A*A
   Induction Step: Prove R^(n+1) = A*A

R^(n+1) = R^n·R = (A*A)*A

$$(A*A)*A = \begin{bmatrix}1&0&0\\1&1&1\\1&0&1\end{bmatrix} * \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&1\\1&0&1\end{bmatrix} = (A*A)$$

Proven

10. (5 points) Ex 10.3, problem 5b

Same conditions as problem 9.

b. Is R reflexive? symmetric? transitive?

Reflexive: Yes
Because (1,1), (2,2) and (3,3) are all 1's in the matrix A
Symmetric: No
Because in A, (1,3) is a 0, while (3,1) is a 1. Therefore (3,1) is in R, but (1,3) is not – so it's not symmetric.
Transitive: No
Because A*A is not a subset of A. (2,1) is a 1 in A*A, but is 0 in A.