CS 1155 Understanding and Constructing Proofs **Fall Quarter, 1995**

Final Examination

December 7

1. **Induction.** Prove by induction that

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

for $r \in \mathbb{R}$, $r \neq 1$, and $n \in N$.

2. **Induction.** Recall that the Fibonacci numbers are defined as follows:

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \quad n \geq 2 \end{aligned}$$

Prove by induction that $f_n < 2^n$ for all $n \geq 1$.

3. **Induction.** Prove by induction that $2^{2n} - 1$ is divisible by 3 for all $n \in N$.

4. **BigOh.** Let $f(n) = n \log n + 4n$ and $g(n) = n^2$. State whether it is true or false that $f(n)$ is $O(g(n))$. Using the *quantification-based* definition of Big-Oh, prove that your answer is correct.

5. **BigOh.** Let f, g and h be arbitrary functions from N to N. Prove that if $f(n)$ is $O(g(n))$ and $g(n)$ is $O(h(n))$, then $f(n)$ is $O(h(n))$. Use only the *quantification-based* definition of Big-Oh.

6. **Proofs.** Prove that if $a + b$ is odd then a is odd or b is odd.

7. **Functions.** Consider $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ given by $f(x, y, z) = (y, x + z)$. Is f one-to-one? If so, prove it; if not, give a counterexample. Is f onto? If so, prove it; if not, give a counterexample.

8. **Logical Implication and Quantification.**

   (a) Is it true that $\neg p \Rightarrow [(p \wedge q) \vee (p \rightarrow q)]$? Prove that your answer is correct.

   (b) For each of the following expressions, state whether it is true or false. Give an informal argument that your answer is correct.

   - $\forall x \; \exists y \; \forall z \; [x + y + z \text{ even}]$
   - $\forall x \; \forall z \; \exists y \; [x + y + z \text{ even}]$

9. **Relations.** The following argument claims to prove that every symmetric and transitive relation is an equivalence relation. What is wrong with the argument? (Do not simply give an example of a relation that is symmetric and transitive, but not reflexive. Instead, point out the *error* in reasoning.)

Let R be symmetric and transitive. To show that R is an equivalence relation, we have to show that R is also reflexive. Because R is symmetric, if $(x, y) \in R$, then $(y, x) \in R$. Because R is transitive, if $(x, y) \in R$ and $(y, x) \in R$, then $(x, x) \in R$. Therefore, R is reflexive.

10. **Relations.** Consider binary relations A and B on set S. For each of the statements below, either prove that it is always true or give a counterexample demonstrating that it can be false. (Recall that $A \setminus B$ is all values in A but not in B.)

    (a) If A and B are symmetric, then $A \setminus B$ is symmetric.

    (b) If A and B are reflexive, then $A \setminus B$ is reflexive.