CS 1155 Understanding and Constructing Proofs **Fall Quarter, 1996**

Midterm Examination 2

December 6

Throughout the exam, the symbol $I\!N$ denotes the natural numbers, and the symbol $I\!R$ denotes the real numbers.

1. **Types.** Let $B = \{-2, -1, 1, 2\}$, $g : B \to B$ where $f(x) = -x$. For each of the following expressions, indicate whether the type of the expression is *set*, *proposition*, *function*, or *ill-formed*:
   (a) $B \subseteq \emptyset$
   (b) $B \times B$
   (c) $g \circ g$
   (d) g is invertible
   (e) $g = O(1)$

2. **Big-Oh.** Let f and g be functions from $I\!N$ to $I\!N$ where $f(n) = 2n^3$ and $g(n) = 1000n^2$. Is $f(n) = O(g(n))$? Prove that your answer is correct, using the logic-based definition of big-Oh.

3. **Big-Oh.** Let f_1, f_2 and g be functions from $I\!N$ to $I\!N$. Show that if f_1 is $O(g)$ and f_2 is $O(g)$ then $f_1 + f_2$ is $O(g)$. (Note that $(f_1 + f_2)(x) = f_1(x) + f_2(x)$).

4. **Functions.** Consider the following informal description of a new function property:

   A function is a "foo function" if and only if every element in the codomain is mapped to by at least two elements in the domain.

   (a) Give a formal logic expression that is true exactly when the function $f : S \to T$ is a foo function.
   (b) Give an example of a foo function, and prove that your function has the foo property.

5. **Functions.** Consider $f : I\!R \times I\!R \to I\!R \times I\!R \times I\!R$ given by $f(a, b, c) = (c, a + b)$.
   (a) Is f one-to-one? If yes, give a proof; if no, give a counterexample.
   (b) Is f onto? If yes, give a proof; if no, give a counterexample.

6. **Sets.** Consider the proposition on sets A, B and C.

   $$(A \cap B) \cup C = A \cap (B \cup C)$$

   (a) Give an example of sets A, B and C such that the proposition is false.
   (b) Give an example of sets A, B and C such that the proposition is true.
   (c) Give a condition that is necessary and sufficient for the proposition to be true. In other words, whenever the condition is true, the proposition is true; whenever the condition is false, the proposition is false.