CS 1155 Understanding and Constructing Proofs **Fall Quarter, 1996**

Final Examination

December 18

Throughout the exam, the symbol $\mathbb{N}$ denotes the natural numbers, and the symbol $\mathbb{R}$ denotes the real numbers.

1. **Types.** Let $A = \{a, b, c, d\}$, $R = A \times A$. For each of the following expressions, indicate whether the type of the expression is *set*, *proposition*, *function*, or *ill-formed*:

   (a) $\{\emptyset\}$
   (b) R is reflexive
   (c) $A \subset R$
   (d) $A \wedge A$
   (e) $a \cap b$

2. **Induction.** For which integers n is $2n + 3 \leq 2^n$? Prove your answer is correct using mathematical induction. What does this tell you about the big-Oh relationship between the functions $2n + 3$ and 2^n?

3. **Big-Oh.** Let f_1, f_2 and g be functions from $\mathbb{N}$ to $\mathbb{N}$. Show that if f_1 is $O(g)$ and f_2 is $O(g)$ then $f_1 \cdot f_2$ is $O(g \cdot g)$. (Note that $(f_1 \cdot f_2)(n) = f_1(n) \cdot f_2(n)$).

4. **Big-Oh.** Let f and g be functions from $\mathbb{N}$ to $\mathbb{N}$ where $f(n) = n \log n$ and $g(n) = n$. Is $f(n) = O(g(n))$? Prove that your answer is correct, using the logic-based definition of big-Oh.

5. **Induction.** Use mathematical induction to prove that

   $$1 \cdot 2 + 2 \cdot 3 + \cdots n(n-1) = n(n+1)(n+2)/3$$

   whenever n is a positive integer.

   (a) Clearly state the proposition $p(n)$ that you are proving.
   (b) State and prove the base case.
   (c) State and prove the inductive step.

6. **Logical Equivalence.** Use a truth table to show that $\neg(p \leftrightarrow q)$ is logically equivalent to $\neg p \leftrightarrow q$.

7. **Equivalence Relations.** Find the smallest equivalence relation (i.e., fewest number of pairs) on the set $\{a, b, c, d, e\}$ that has the relation $\{(a, b), (d, e)\}$ as a subset.

8. **Relations.** Consider a new property of a relation, one called the "anti-transitive" property. Informally, a relation is anti-transitive if there is no way to short-cut a path containing two edges by taking a direct edge.

(a) Give a formal logic expression that is true exactly when the relation R on the set S is anti-transitive.

(b) Give an example of an anti-transitive relation, and prove that your relation is anti-transitive.

(c) Is it possible for a relation to be both transitive and anti-transitive? If yes, give an example; if no, give a brief argument why.

9. **Proof Methods.** Prove that if n^3 is a multiple of 3 then n is a multiple of 3.

10. **Proof Methods.** Prove that $\sqrt[3]{3}$ is irrational. You may use the lemma from the previous problem (even if you are unable to prove it).