CS 1155 Understanding and Constructing Proofs **Fall Quarter, 1997**

Final Examination

December 9

Throughout the exam:

- N denotes the natural numbers, $\{0, 1, 2, ...\}$
- R denotes the real numbers
- Z denotes the integers, $\{..., -2, -1, 0, 1, 2, ...\}$

1. Assume that A, B and C are arbitrary sets with elements from the universe $U = N$. Prove or disprove:

$$((A \cap B \neq \emptyset) \wedge (B \cap C \neq \emptyset) \wedge (A \cap C \neq \emptyset)) \rightarrow (A \cap B \cap C \neq \emptyset)$$

2. Let U be the universe of all people. Let $T(A, B)$ be the propositional function A trusts B. Using quantifiers and the propositional function T, express the following English sentences:
   (a) Gary trusts everybody.
   (b) Nobody trusts Bob. (not even himself!)
   (c) Alice only trusts herself.
   (d) There are two (distinct) people who trust each other.
   (e) Carol trusts everyone trusted by David.
   (f) Write in English and in mathematical notation the negation of sentence (e).

3. Let $f : N \rightarrow N$ be the function $f(n) = 2n^2 + 3n + 8$ and $g : N \rightarrow N$ be the function $g(n) = n^2$. Prove that f is big-Oh g.

4. Let $f : N \rightarrow N$ be the function $f(n) = 4^n$ and $g : N \rightarrow N$ be the function $g(n) = 16 \times 2^n$. Prove that f is not big-Oh g.

5. Prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

6. Suppose you have two types of coins, 3 cent coins and 4 cent coins.
   (a) What total values can be formed using these two types of coins?
   (b) Prove that your answer to part (a) is correct.

7. Let $f : N \times N \rightarrow N \times N$ be defined by $f(x, y) = (y, 2x + y)$.
   (a) Prove or disprove that f is one-to-one.
   (b) Prove or disprove that f is onto.

8. Prove the following:
   (a) For any integer k, if k is odd then $3k$ is odd.

(b) For any even integer n, if n is a multiple of 3, then n is a multiple of 6. You may use the lemma stated in part (a).

9. Let $f : N \rightarrow N$ be defined by $f(n) = 2n$. Let $A = \{a, b, c\}$. State whether each of the following is of type *set*, *function*, *proposition*, *ill-formed*.

   (a) $f = O(n)$

   (b) f has an inverse

   (c) $\mathcal{P}(\mathcal{A}) \cap \mathcal{A}$

   (d) $\emptyset \in \mathcal{P}(\mathcal{A})$

   (e) $f \circ f$

10. Let $f : N \times N \rightarrow N$ be defined by $f(i, j) = ij(i + j)$. Let A be the set of odd integers. Prove that the intersection of the range of f and the set A is the empty set. (Formally, $range(f) \cap A = \emptyset$.)