CS 1155 Understanding and Constructing Proofs **Summer Quarter, 1996**

Final Examination

September 30

Throughout the exam, the symbol $\mathbb{N}$ denotes the natural numbers, the symbol $\mathbb{R}$ denotes the real numbers and the symbol Z denotes the positive and negative integers.

1. **Big-Oh.** Let f and g be functions from $\mathbb{N}$ to $\mathbb{N}$ where $f(n) = 4n$ and $g(n) = 2n - 10$. Prove that $f(n)$ is $O(g(n))$, using the logic-based definition of big-Oh.

2. **Big-Oh.** Let f and g be functions from $\mathbb{N}$ to $\mathbb{N}$ where $f(n) = 4^n$ and $g(n) = 3^n$. Prove that $f(n)$ is not $O(g(n))$, using the logic-based definition of big-Oh.

3. **Functions.** Let $f : R \to R$ be given by $f(x) = ax + b$, where $a, b \in R$ and $a \neq 0$.

   (a) If f one-to-one? If yes, give a proof; if no, give a counterexample.

   (b) Is f onto? If yes, give a proof; if no, give a counterexample.

   (c) Does f have an inverse? If yes, give the inverse.

   (d) State how the answers above differ if $a = 0$. (Proofs are not required.)

4. **Relations.** Suppose that R and S are relations on a set A. Prove or disprove the following:

   (a) R is not symmetric and S is not symmetric implies $R \cup S$ is not symmetric.

   (b) R is reflexive and S is reflexive implies $R \setminus S$ is anti-reflexive.

5. **Proofs.** We can formally express the proposition "n is even" by stating that $\exists x \in Z n = 2x$. Use this formal definition to prove the following: if a^3 is even, then a is even. (Hint: consider the contrapositive.)

6. **Types.** Let $A = \{1, 2, 3\}$, $f : A \to A$ where $f(x) = x$, and $R = A \times A$. For each of the following expressions, indicate whether the type of the expression is *set*, *proposition*, *function*, or *ill-formed*:

   (a) f is $O(1)$

   (b) $[1]$

   (c) $[1] = R^{\leftarrow}$

   (d) $f o f$

   (e) $2 \sim 3$, where $x \sim y$ is defined by $x < y$

7. **Induction.** Find (guess) a formula for the sum of the first n even positive integers. Use induction to prove that your formula is correct.

8. **Induction.** Use induction to prove that $2^n < n^2$ for any positive integer $n > 4$.

9. **Induction.** Use induction to show that any value greater than 7 cents can be formed using only 3-cent and 5-cent coins. Clearly state the proposition $p(n)$ that you are proving.

10. **Equivalence Relations.** Find the smallest equivalence relation (i.e., fewest number of pairs) on the set $\{a, b, c, d, e\}$ containing the relation $\{(a, b), (d, e)\}$.

11. **Logic.** Is $(p \wedge q) \rightarrow (p \rightarrow q)$ a tautology? Prove that your answer is correct.