Every assignment will be due at the beginning of class. Recall that you can collaborate in groups and/or use external references, but you must acknowledge the group/references that you used, and you must *always write your solutions alone*. Remember that for 90% of the people, more than 50% of the understanding happens during writing/implementation/etc. (And this is not true only for CS 1050. It is true for mostly everything, at least technical).

Please read the entire homework before starting to work on it. This homework refers to material covered in Lectures II though V. You may want to refer to the readings assigned during these Lectures, and to to the Lecture Outlines. All are posted on the class website.

Please stop by for questions during office hours of instructor or TAs and send email to mihail@cc.gatech.edu with title 1050 at any time. This helps you, but it also helps us! Sometimes it helps us understand where the class stands and where we should put more or less emphasis. And sometimes, you give us presentational and technical ideas that we would have not thought of otherwise. So keep all communication links open!

Please print this document, and write your solutions on the printout. Please hand-in the completed printout.

PRINT YOUR NAME HERE:............................... MILENA MIHAIL

WRITE YOUR EMAIL HERE:.............................. mihail@cc.gatech.edu
Problem 1: (10 points)
Which of the following statements are propositions, and which are not propositions. For each statement that is a proposition, write (without explanation) if it is true or false. For each statement that is not a proposition, give a one line explanation why it is not a proposition.

(i) \(2 + 2 = 4\)  
**True Proposition**

(ii) \(2 + 7 = 10\)  
**False Proposition**

(iii) \(\forall x \forall y (x + y = y + x)\), where \(x\) and \(y\) are real numbers.  
**True Proposition**

(iv) \(\forall x \exists y (x + y = y + x)\), where \(x\) and \(y\) are real numbers.  
**True Proposition**

(v) If the earth is flat, then \(2 + 2 = 7\).  
**True Proposition** (False \(\Rightarrow\) anything is true!) "the earth is flat"

(vi) Go directly to jail.  
Not a proposition. Does not admit a true/false value

(vii) Your place or mine?  
Not a proposition. Does not admit a true/false value

(viii) \(3 = x + 7\).  
Not a proposition. Does not admit a definite true/false value. Its true/false value depends on \(x\)

(ix) If \(2 + 2 = 4\) then \(2 + 3 = 5\).  
**True Proposition**

(x) If \(2 + 2 = 5\) then \(2 + 3 = 6\).  
**True Proposition**

(*) A lot of students get confused about (v) and (x). The point is to recall the truth table of \(p \Rightarrow q\):

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Notice: \(p \Rightarrow q\) is false if and only if \(p\) is true but \(q\) is false. If \(p\) is false then \(p \Rightarrow q\) is true. Like "the earth is flat," or \(2 + 2 = 5\) then \(p \Rightarrow q\) is true.
Problem 2: (10 points)
The statement $x - y = y - x$, where $x$ and $y$ are real numbers is not a proposition. This is because its truth value depends on $x$ and $y$. For example, if $x = y = 0$ then $x - y = y - x$, while if $x \neq 0 \neq y$ then $x - y \neq y - x$. To turn the statement $x - y = y - x$ to a proposition, we have to bind the variables $x$ and $y$ by existential and/or universal quantifiers. For each of the four ways of binding $x$ and $y$ that follow, state if the corresponding proposition is true or false. Give a short explanation. If you find some cases somewhat hard or unclear, try to do Problem 3 (or part of it) first.

(i) $\forall x \forall y \ (x - y = y - x)$.
Simplification: $\forall x \forall y \ x = y$
Clearly false. It says that all $x$ and $y$ are equal!
But they are not! Counter-example: if $x = 0$, $y = 1$ then $x \neq y$

(ii) $\forall x \exists y \ (x - y = y - x)$.
Simplification: $\forall x \exists y \ x = y$
True. It says that for all $x$ there exists a $y$ such that $x = y$.
If we set $y := x$, then this is such a $y$.

(iii) $\exists x \forall y \ (x - y = y - x)$
Simplification: $\exists x \forall y \ x = y$
False. It says that there exists a number $x$
such that all other numbers $y$ are equal to $x$.

(iv) $\exists x \exists y \ (x - y = y - x)$
Simplification: $\exists x \exists y \ x = y$
Clearly true.
Example. if $x = 3$ and $y = 3$
then $x = y = 3$.

How can we explain "formally" that this is false?
We will instead explain that its negation is true.
But its negation is (Problem 3) $\forall x \exists y \ x \neq y$
which says that for every number $x$, there exists a number $y$
which is not equal to $x$,
which is clearly true.
Problem 3: (10 points)

In Problem 3, statements (i) through (iv) are propositions. Therefore, their negations are also propositions. For each of the propositions (i) through (iv) in Problem 3, write clearly its negation, where the negation sign $\neg$ does not appear anywhere in the proposition.

For example, for (i), the negation of $\forall x \forall y \ (x - y = y - x)$ is $\neg[\forall x \forall y \ (x - y = y - x)]$.
This is equivalent to $\exists x \neg[\forall y \ (x - y = y - x)]$.
This is equivalent to $\exists x \exists y \neg(x - y = y - x)$.
This is equivalent to $\exists x \exists y \ (x - y \neq y - x)$.
You may even go one step further, and observe that $(x - y \neq y - x) \iff 2x \neq 2y \iff x \neq y$.
You then get the cleanest form: $\exists x \exists y \ x \neq y$.

\[ (\text{ii}) \quad \neg \left[ \forall x \exists y \ (x = y) \right] \quad \text{equivalent to} \]
\[ \exists x \left[ \neg \left( \exists y \ (x = y) \right) \right] \quad \text{equivalent to} \]
\[ \exists x \forall y \neg (x = y) \quad \text{equivalent to} \]
\[ \exists x \forall y \quad x \neq y \]

\[ (\text{iii}) \quad \neg \left[ \exists x \forall y \ (x = y) \right] \quad \text{equivalent to} \]
\[ \forall x \left[ \neg \left( \forall y \ (x = y) \right) \right] \quad \text{equivalent to} \]
\[ \forall x \exists y \neg (x = y) \quad \text{equivalent to} \]
\[ \forall x \exists y \quad x \neq y \]

\[ (\text{iv}) \quad \neg \left[ \exists x \exists y \ (x = y) \right] \quad \text{equivalent to} \]
\[ \forall x \left[ \neg \left( \exists y \ (x = y) \right) \right] \quad \text{equivalent to} \]
\[ \forall x \forall y \neg (x = y) \quad \text{equivalent to} \]
\[ \forall x \forall y \quad x \neq y \]
Problem 4: (10 points)

Using truth tables, prove the following tautologies (where $p$ and $q$ are propositional variables):

(i) $[p \land (p \Rightarrow q)] \Rightarrow q$ (aka Modus Ponens or Direct Proof).

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<th>$p \Rightarrow q$</th>
<th>$p \land (p \Rightarrow q)$</th>
<th>$[p \land (p \Rightarrow q)] \Rightarrow q$</th>
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(ii) $[(p \Rightarrow q) \land \neg q] \Rightarrow \neg p$ (aka Modus Tollens or Contradiction).

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(iii) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$ (aka Contraposition).

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(iv) $(p \Rightarrow q) \Leftrightarrow (\neg p \lor q)$.

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Problem 5: (10 points)

Express these systems specifications using the propositions $p$ "The message is scanned for viruses" and $q$ "The message was sent by an unknown system" together with logical connectives.

(a) "The message is scanned for viruses whenever the message was sent from an unknown system".
(b) "The message was sent by an unknown system but it was not scanned for viruses".
(c) "It is necessary to scan the message for viruses whenever it was sent from an unknown system".
(d) "When a message is not sent from an unknown system it is not scanned for viruses".

Show that the above specifications are not consistent, and explain it also in English. Then show that if you drop exactly one of (a), (b), (c), or (d), then the system will become consistent, thus completely identifying the cause of inconsistency.

(a) $q \implies p$

(b) $q \land \neg p$

(c) $q \implies p$

(d) $\neg q \implies \neg p$

First of all, I can forget (c) because it is the same as (a).

So it suffices to check:

$q \implies p$, $q \land \neg p$ and $\neg q \implies \neg p$

for consistency.
Fast answer:

Quickly, by inspection, I can see that \( q \Rightarrow p \) and \( q \land \neg p \) are not consistent.

In English, this means that while a requirement says \( q \Rightarrow p \) i.e. if the sender is unknown the message should be checked for viruses, it was also the case that \( q \land \neg p \) i.e. a sender was unknown and the message was not checked for viruses.

If I drop (b), then \( q \Rightarrow p \) and \( \neg q \Rightarrow \neg p \) are consistent and equivalent to \( q \iff p \).
There are the system specifications.

There is no line where all of them are simultaneously true.
The system is not consistent.
If I drop (b) \( q \lor \neg p \) then I get a consistent system:

\[
\begin{array}{ccc|c}
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\end{array}
\]

These are the system specifications. They are simultaneously satisfied when \( p = q = T \) and when \( p = q = F \).
Problem 6: (10 points)

A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so it the cook; the cook and the gardener cannot both be telling the truth; the cook and the gardener are not both lying; and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning.

We translate the information to logic:

\[ B = \text{butler is saying the truth} \]
\[ C = \text{cook} \]
\[ G = \text{gardener} \]
\[ H = \text{handyman} \]

We also have:

\[ B \implies C \]  
\[ \neg (C \land G) \]  
\[ \neg (\neg C \land G) \equiv C \lor G \]  
\[ H \implies \neg C \]

(if the butler is saying the truth then so is the cook)

(the cook and gardener cannot both be telling the truth)

(the cook and the gardener are not both lying)

(or else, one of the cook or gardener is telling the truth)

(if the handyman is telling the truth then the cook is lying)
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Everything empty is true.

We know all columns here must be true, so we only need to look at lines "→"
Here is the status of B, C, G, H for these four critical lines:

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There is no column (witness) where in all four critical lines the status of a witness is consistent. I.e. either everywhere T or everywhere F.

Therefore no inference can be made.
Problem 7: (10 points)

(iA) Give a direct proof (one line argument) that the product of two consecutive integers is always an even number.

One of the integers will be even, which makes the product even.

(iB) Give a direct proof that, for any integer $k$, $k^2 + k$ and $k^2 - k$ are both even numbers.

Let $k$ be any integer.

Notice that $k^2 + k = k(k+1)$

and $k^2 - k = k(k-1)$.

In either case we are talking about the product of two consecutive integers, which is always even.

by (iA)
(iiA) Prove that, if \( n \) and \( m \) are odd integers, then the product \( n \times m \) is also an odd integer. A simple direct proof should suffice.

\[
\text{If } n \text{ is odd it is of the form } n = 2k + 1 . \\
\text{If } m \text{ is odd it is of the form } m = 2l + 1 .
\]

Now \( n \times m = (2k + 1)(2l + 1) \)

\[
= 4kl + 2k + 2l + 1
\]

\[= \text{EVEN} + 1\]

\[= \text{ODD} \]

(iiB) Prove that, if \( n \) and \( m \) are integers and the product \( n \times m \) is an odd integer, then both \( n \) and \( m \) are odd integers. You may need to use the contrapositive principle for this proof.

\( p = \text{"} m \times m \text{ is odd"} \)
\( q = \text{"} m \text{ is odd and } n \text{ is odd"} \)

We need to show \( p \implies q \)

The equivalent contrapositive is \( \neg q \implies \neg p \)

So we will equivalently show \( \neg q \implies \neg p \)

\( \neg q = \text{"} \text{at least one of } m \text{ or } n \text{ is even"} \)
\( \neg p = \text{"} \text{m} \times n \text{ is even"} \)

So we need to show that if at least one of \( m \) and \( n \) is even, then \( m \times n \) is even.

But this is obviously true.
Problem 8: (10 points)

Prove that $\sqrt{3}$ is an irrational number. You may need to use an argument by contradiction.

Assume for the purposes of contradiction that

\[ \sqrt{3} = \frac{p}{q}, \quad \text{where } p \text{ and } q \text{ are relatively prime integers.} \]

Then

\[ 3 = \frac{p^2}{q^2} \quad \Rightarrow \quad 3q^2 = p^2 \quad \ast \]

**Case 1**  
$p$ and $q$ both even, **impossible** because then they are not relatively prime.

**Case 2**  
$p = 2k$, $q = 2l + 1$.

Then $\ast$ implies:

\[ 3(2l+1)^2 = (2k)^2 \quad \Rightarrow \]
\[ 3(4l^2 + 4l + 1) = 4k^2 \quad \Rightarrow \]
\[ 12l^2 + 12l + 3 = 4k^2 \quad \Rightarrow \]
\[ \text{EVEN} + 1 = \text{EVEN} \quad \Rightarrow \]
\[ \text{ODD} = \text{EVEN}, \quad \text{impossible} \]

**Case 3**  
$p = 2k + 1$, $q = 2l$.

Then $\ast$ implies:

\[ 3(2l)^2 = (2k+1)^2 \quad \Rightarrow \]
\[ 3 \cdot 4l^2 = 4k^2 + 4k + 1 \quad \Rightarrow \]
\[ \text{EVEN} = \text{EVEN} + 1 \quad \Rightarrow \]
\[ \text{EVEN} = \text{ODD}, \quad \text{impossible} \]
Case 4: \( p = 2k + 1, \quad q = 2k + 1 \)

Then \( \ast \) implies:

\[
3(2k+1)^2 = (2k+1)^2 \implies 3(4k^2 + 4k + 1) = 4k^2 + 4k + 1 \implies
\]

\[
12k^2 + 12k + 3 = 4k^2 + 4k + 1 \implies
\]

\[
12k^2 + 12k + 2 = 4k^2 + 4k \implies
\]

\[
6k^2 + 6k + 1 = 2k^2 + 2k \implies
\]

\[
\frac{6k^2 + 6k + 1}{2} = \text{ Even}
\]

\[
\text{Even} + 1 = \text{ Even} \implies
\]

\[
\text{Odd} = \text{ Even}, \quad \text{impossible}
\]

All cases lead to a contradiction, therefore the assumption \( \sqrt{3} = \frac{p}{q} \) is false.
Problem 10: (10 points)

Prove that every odd integer is the difference of two squares.

Hint: The proof is direct and very simple. But how does one start thinking about it? Examples are many times a good place to start, since they might indicate a pattern. In this case, try to express 1, 3, 5, 7, 9 as the difference of two squares, and you immediately see the pattern. For example, for 3, it is obvious that \(3 = 4 - 1 = 2^2 - 1^2\). Also 5 is easy: \(5 = 9 - 4 = 3^2 - 2^2\). Now keep going...

\[
\begin{align*}
\text{Thinking:} & \quad 3 &= 4 - 1 = 2^2 - 1^2 \\
& \quad 5 &= 9 - 4 = 3^2 - 2^2 \\
& \quad 7 &= 16 - 9 = 4^2 - 3^2 \\
& \quad 9 &= 25 - 16 = 5^2 - 4^2 \\
\text{odd integer of the form} & \quad 2k - 1 \\
\text{appears to be as} & \quad k^2 - (k - 1)^2
\end{align*}
\]
So I will prove the following stronger fact:

**Fact**

Every odd integer, which is therefore a number of the form $2k-1$ for some integer $k$, can be written as: $2k-1 = k^2 - (k-1)^2$

**Proof**

I need to verify that $2k-1 = k^2 - (k-1)^2$

equivalently, $2k-1 = k^2 - (k^2 + 2k + 1)$
equivalently, $2k-1 = k^2 - k^2 + 2k - 1$
equivalently, $2k-1 = 2k-1$

Obviously true.