9-16-08
CS 1050 A

Homework 4
Due: 9-23-08

Every assignment will be due at the beginning of class. Recall that you can collaborate in groups and/or use external references, but you must acknowledge the group/references that you used, and you must *always write your solutions alone.* Remember that for 90% of the people, more than 50% of the understanding happens during writing/implementation/etc. (And this is not true only for CS 1050. It is true for mostly everything, at least technical).

Please read the entire homework before starting to work on it. This homework refers to material mainly covers induction, structural induction and strong induction.

Please stop by for questions during office hours of instructor or TAs and send email to mihail@cc.gatech.edu with title 1050 at any time. This helps you, but it also helps us! Sometimes it helps us understand where the class stands and where we should put more or less emphasis. And sometimes, you give us presentational and technical ideas that we would have not thought of otherwise. So keep all communication links open!

Please print this document, and write your solutions on the printout.
Please hand-in the completed printout.

PRINT YOUR NAME HERE:.........MILENA MIHAIL................

WRITE YOUR EMAIL HERE:...........mihail@cc.gatech.edu................

Problem 1: (25 points)

(a1) Prove by induction that, for all positive integers n, 6 divides $n^3 - n$.
(a2) Can you also give a direct proof?

(a1) ANSWER:
Proof inductive on n.
Base Case: Check that the proposition is true for $n = 1$. But, for $n = 1$, $n^3 - n = 1^3 - 1 = 0$, which is clearly divisible by 6.
Inductive Hypothesis: Assume that, for some $n = k \geq 1$, $k^3 - k$ is indeed divisible by 6, or equivalently, $k^3 - k$ is a multiple of 6, or equivalently, for some integer $q = q_k$,

$$k^3 - k = 6q_k \ .$$

Inductive Step: We want to show that for, $n = k + 1$, $(k + 1)^3 - (k + 1)$ is a multiple of 6.
As always, we have to *express the quantity involved in the case of* $n = k+1$ *in terms of the quantity involved in the case of* $n = k$, for which we can use the induction hypothesis. We therefore write:

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3(k^2 + k) \\ &= (k^3 - k) + 3k(k+1)\end{aligned}$$

Now, from the induction hypothesis, we know that $k^3 - k = 6q_k$. Also, $k(k + 1)$ is the product of two consecutive integers, so one of them must be even, so $k(k + 1) = 2p_k$, for some integer p_k. Therefore, we get,

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 - k) + 3k(k+1) \\ &= 6q_k + 3 \times 2p_k \\ &= 6q_k + 6p_k \ .\end{aligned}$$

(a2) ANSWER:
Direct proof. General principles: (I) Exactly one of three consecutive numbers is a multiple of 3. (II) The integer before and after an odd number is even.
Now notice: $n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1)$.
Case 1: Suppose that n is a multiple of 3.
Case 1a: Suppose n is also even. Then n is a multiple of 6, and so $n(n - 1)(n + 1)$ is a multiple of 6.
Case 1b: Suppose that n is odd. Then both $(n - 1)$ and $(n + 1)$ are even, so $n(n - 1)(n + 1)$ is a multiple of 6.
Case 2: Suppose that either $(n - 1)$ or $(n + 1)$ is a multiple of 3.
Case 2a: Suppose the above number is also even. Then it is a multiple of 6, and so $n(n - 1)(n + 1)$ is a multiple of 6.
Case 2b: Suppose that the above number is odd. Then n must be even, so $n(n - 1)(n + 1)$ must be a multiple of 6.

(b1) Give a direct proof that, whenever n is an odd integer, $n^2 - 1$ is divisible by 8.
(b2) Prove by induction that, whenever n is an odd integer, $n^2 - 1$ is divisible by 8.

(b1) ANSWER:
$n^2 - 1 = (n-1)(n+1)$.
Now, if n is odd, then $n = 2k+1$ for some integer k.
Therefore $n - 1 = (2k+1) - 1 = 2k$ and $n + 1 = (2k+1) + 1 = 2(k+1)$.
So $n^2 - 1 = 2k \times 2(k+1) = 4k(k+1)$, for some integer k.
But, for every integer k, $k(k+1)$ is the product of two consecutive integers, one of which must be even, so $k(k+1) = 2l$, for some integer l.
Therefore, $n^2 - 1 = 4 \times 2l$, for some integer l, therefore $n^2 - 1 = 8l$.

(b2) ANSWER:
We may assume that $n > 0$, since the proposition involves only n^2 which is always positive.
Base Case $n = 1$, we can check that $n^2 - 1 = 1^2 - 1 = 0$, which is divisible by 8.
Inductive Hypothesis:
Assume that, for odd integer $n = k \geq 1$, $k^2 - 1 = 8q_k$.
Inductive Step: We want to show that, for odd integer $n = k+2$, $(k+2)^2 - 1 = 8q_{k+2}$. As always, we have to *express the quantity involved in the case of* $n = k+2$ *in terms of the quantity involved in the case of* $n = k$, for which we can use the induction hypothesis. We therefore write: $(k+2)^2 - 1 = k^2 + 4k + 4 - 1 = (k^2 - 1) + 4(k+1)$.
But, by the inductive hypothesis, $k^2 - 1 = 8q_k$.
Also, since k is odd, $(k+1)$ must be even, so $(k+1) = 2l$, for some integer l.
We may therefore write $(k+2)^2 - 1 = 8q_k + 4 \times 2l = 8q_k + 8l$, which is obviously a multiple of 8.

(c) Prove by induction that, for all positive integers n, $1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

ANSWER: Inductuve on n.
Base Case: For $n = 1$, it is indeed true that $1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 = 1$.
Inductive Hypothesis: Assume that, for $n = k \geq 1$, it is true that

$$1^3 + 2^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

. Inductive Step: We want to show that, for $n = k + 1$,

$$1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

. As always, we have to *express the quantities involved in the case of* $n = k + 1$ *in terms of the quantities involved in the case of* $n = k$, for which we can use the induction hypothesis. We therefore write:

$$\begin{aligned} 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 &= \left(1^3 + 2^3 + \ldots + k^3\right) + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \text{ , by the ind. hyp.} \end{aligned}$$

It therefore suffices to verify that

$$\begin{aligned} \left(\tfrac{k(k+1)}{2}\right)^2 + (k+1)^3 &= \left(\tfrac{(k+1)(k+2)}{2}\right)^2 && \Longleftrightarrow \\ k^2(k+1)^2 + 4(k+1)^3 &= (k+1)^2(k+2)^2 && \Longleftrightarrow \\ k^2 + 4(k+1) &= (k+2)^2 && \Longleftrightarrow \\ k^2 + 4k + 4 &= k^2 + 4k + 4 && \text{obviously true.} \end{aligned}$$

Problem 2: (25 points)

(a1) Prove that a non-negative integer is divisible by 9 if and only if the sum of its digits (in decimal representation) is divisible by 9.
(a2) Argue that the following recursive algorithm computes the remainder of the division of a non-negative integer by 9: Let x be the initial number in decimal representation. Let y be the sum of all the digits of x. If $y < 10$ then output y, else $x := y$ and recurse.

(a1) ANSWER:
Let $x = x_n x_{n-1} \ldots x_2 x_1 x_0$ be the decimal representation of x. We thereore have :

$$\begin{array}{rcllllllllllll} x & = & x_n & 10^n & + & x_{n-1} & 10^{n-1} & + & \ldots & + & x_2 & 100 & + & x_1 \quad 10 \quad +x_0 \\ & = & x_n & (\underbrace{99\ldots9}+1) & + & x_{n-1} & (\underbrace{99\ldots9}+1) & + & \ldots & + & x_2 & (99+1) & + & x_1 \quad (9+1) \quad +x_0 \\ & = & x_n & (\overbrace{n \text{ 9's}}+1) & + & x_{n-1} & (\overbrace{(n-1) \text{ 9's}}+1) & + & \ldots & + & x_2 & (99+1) & + & x_1 \quad (9+1) \quad +x_0 \end{array}$$

Now the right hand side of the above can be rearranged as:

$$\begin{array}{llllll} x_n & \underbrace{99\ldots9} & + & x_{n-1} & \underbrace{99\ldots9} & +\ldots+x_2 99+x_1 9+ \quad (x_n+x_{n-1}+\ldots+\ldots x_2+x_1+x_0) \\ x_n & \overbrace{n \text{ 9's}} & + & x_{n-1} & \overbrace{(n-1) \text{ 9's}} & +\ldots+x_2 99+x_1 9+ \quad (x_n+x_{n-1}+\ldots+\ldots x_2+x_1+x_0) \end{array}$$

Now, clearly, each of the numbers $99\ldots9$ is a multiple of 9, therefore, x can be written as

$$x = 9\kappa + (x_n + x_{n-1} + \ldots + \ldots x_2 + x_1 + x_0) \tag{1}$$

Clearly, by Equation (1) above, x is divisible by 9 if and only if the sum of the digits of x $\sum_{i=0}^{n} x_i$ is divisible by 9.

(a2) ANSWER: More generally, by Equation (1) above, the remainder of x when divided by 9 is equal to the remainder of $y = \sum_{i=0}^{n} x_i$ when divided by 9 (since, of course, the remainder of 9κ when divided by 9 is zero).
If $y = \sum_{i=0}^{n} x_i$ is a one digit number, then its remainder when divided by 9 is obvious. In particular, if $y = 9$ then its remainder when divided by 9 is 0, while if $y \neq 9$ (and since $y > 0$) then its remainder when divided by 9 is y itself.
But $y = \sum_{i=0}^{n} x_i$ is not necessarily a one digit number. In fact, in general, y might be large enough that one cannot tell its remainder when divided by 9 by inspection. This is because we are considering the general case where x has n digits, where n can be arbitrarily large. Obviously, adding n single digit x_i's might end up with a large y. For example, let us say $x = 9278765978767548797688382567$. Then $y = 9+2+7+8+7+6+5+9+7+8+7+6+7+5+4+8+7+9+7+6+8+8+3+8+2+5+6+7 = 181$. We know that x and y have the same remainder when divided by 9, but it is still not easy to tell what this remainder is because $y = 181$ is still somewhat large.
The main point now is is that we can repeat the same procedure to y, by adding up y's digits. We know that as long as $y < x$ (which is quite obvious in this example), we will be making progress. So let's add the digits of $y = 181$ and get $z = 1 + 8 + 1 = 10$. We know that the remainder of z when divided by 9 is the same as the remainder of y when divided by 9, which is the same as the remainder of x when divided by 9. Once again, $z = 10$ is not a single digit number, and so its remainder when divided by 9 is the same as the remainder of the sum of its digits, which in this case is $r = 1 + 0 = 1$. Now this is indeed a single digit number. So we conclude that the remainder of x when divided by 9 is 1.

Finally, to ensure that this process works in general, we have to argue one point:
Claim: For any $x = x_n x_{n-1} \ldots x_2 x_1 x_0$ in decimal representation, $n \geq 1$, the sum of its digits $y = \sum_{i=0}^{n} x_i$ satisfies $y < x$.
Proof of Claim: First, we can verify that the claim is true for all two digit numbers x. So we will be concerned with the case where $n \geq 2$. Since x is an n-digit number, we know that $x_n \geq 1$, therefore $x \geq x_n 10^n$.
On the other hand, the largest that $y = \sum_{i=0}^{n} x_i = x_n + \sum_{i=0}^{n-1} x_i$ can get is $y \leq x_n + 9n$ (when all x_i's, except for x_n, are 9.
We now only need to verify that $x_n + 9n < x_n 10^n$, for all $n \geq 2$, and for any $1 \leq x_n \leq 9$.
Equivalently, we need to verify that $9n < x_n(10^n - 1)$, for all $n \geq 2$, and for any $1 \leq x_n \leq 9$.
We can prove this last proposition by induction on n.
To avoid confusion between the n of the induction and the digit x_n, we will equivalently show that $9n < \alpha(10^n - 1)$, for all $n \geq 2$, and for any $1 \leq \alpha \leq 9$.
Base Case: For $n = 2$, the left hand side is 18, and the right hand side is $99\alpha \geq 99$. So the left hand side is strictly smaller than the right hand side.
Inductive Hypothesis: For $n = k \geq 2$, assume that, for all $1 \leq \alpha \leq 9$, $9k < \alpha(10^k - 1)$ is true.
Inductive Step: For $n = k + 1$, we want to show that, for all $1 \leq \alpha \leq 9$, $9(k + 1) < \alpha(10^{k+1} - 1)$.
But, by the inductive hypothesis, we know:

$$9k < \alpha(10^k - 1)$$

and by assumption of α we know that

$$9 < 10\alpha$$

Adding the above inequalities we get:

$$\begin{array}{rcll} 9k + 9 & < & \alpha(10^k - 1) + 10\alpha & \iff \\ 9(k+1) & < & \alpha 10^k + 10\alpha - \alpha & \iff \\ 9(k+1) & < & \alpha 10^{k+1} - \alpha & \iff \\ 9(k+1) & < & \alpha(10^{k+1} - 1) & \end{array}$$

(b1) Let xy be a two digit non-negative integer in decimal notation, that is, $xy = x10 + y$. Prove that, for any $0 \leq x, y \leq 9$, $xy + yx$ is a multiple of 11.

(b1) ANSWER:
Direct proof. General principal: interpret the decimal representation.

$$xy = 10x + y.$$

$$yx = 10y + x.$$

Therefore, by adding the above, we get

$$\begin{aligned} xy + yx &= (10x + y) + (10y + x) \\ &= (10x + x) + (10y + y) \\ &= 11x + 11y \\ &= 11(x + y) \end{aligned}$$

(b2) Let us generalize: Let $a_{n-1}a_{n-2}\ldots a_1a_0$ be an n digit non-negative integer in decimal notation, where n is an even number. That is, $a_{n-1}a_{n-2}\ldots a_1a_0 = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \ldots + a_1 10 + a_0$. Prove that, $a_{n-1}a_{n-2}\ldots a_1a_0 + a_0a_1\ldots a_{n-2}a_{n-1}$ is a multiple of 11. Hint: Realize that 1001 is a multiple of 11 (since 1001=990+11), 100001 is a multiple of 11 (since 100001=99990+11), and so on.

(b2) ANSWER:
Let us first prove the hint:
Claim: Any number whose decimal representation is 1, followed by an even number of 0's, followed by 1, is a multiple of 11. In particular, if there are $2n$ 0's, let us call this number q_n.
Proof: Let us further call p_n the number whose decimal representation consists of $2n$ 9's followed by a zero. We can immediately see that this number is a multiple of 11:

$$p_n = 99 \times 10^{2n-1} + 99 \times 10^{2n-3} + \ldots + 990$$

. And we can immediately see that $q_n = p_n + 11$, thus q_n is a multiple of 11.

Now let us look at

$$\begin{array}{ccccccc}
a_{n-1}10^{n-1} & + & a_{n-2}10^{n-2} & + \ldots + & a_1 10 & + \\
 & & & + & & \\
a_0 10^{n-1} & + & a_1 10^{n-2} & + \ldots + & a_{n-2}10 & + \\
 & & & = & & \\
(a_{n-1}+a_0)10^{n-1} & + & (a_{n-2}+a_1)10^{n-2} & + \ldots + & (a_1+a_{n-2})10 & + \\
 & & & = & & \\
(a_{n-1}+a_0)(10^{n-1}+1) & + & (a_{n-2}+a_1)(10^{n-2}+10) & + \ldots + & (a_{(n+3)/2}+a_{(n-3)})/2)1001 & + \\
 & & & = & & \\
(a_{n-1}+a_0)q_{(n+1)/2} & + & (a_{n-2}+a_1)q_{(n-1)/2} \times 10 & + \ldots + & (a_{(n+3)/2}+a_{(n-3)})/2)1001 \times 10^{(n-3)/2} & + & (a_{(n+}
\end{array}$$

whish is a multiple of 11, since each term involves a q_i.

Problem 3: (25 points)

(a) Prove, by induction, that the recurrence $T(n) = 3T(\frac{n}{3}) + n$ with $T(1) = 0$ solves to $T(n) = n \log_3 n$. You may assume that n is a power of 3.

ANSWER:
Since n is a power of 3, let us denote $n = 3^N$, and note that $log_3 3^N = N$. We now need to show:
The recurrence $T(3^N) = 3T(3^{N-1}) + 3^N$, with $T(3^0) = 0$, solves to $T(3^N) = 3^N N$.
Proof inductive on N.
Base Case: For $N = 0$, the expressions $T(3^0) = 0$ and $T(3^0) = 3^0 0$ both evaluate to 0.
Inductive Hypothesis: Assume that, for $N = k \geq 0$,it is indeed the case that

$$T(3^k) = 3^k k \ .$$

Inductive Step: We want to show that, for $N = k + 1$,

$$T(3^{k+1}) = 3^{k+1}(k + 1) \ .$$

But, by the form of the recurrence, we have:

$$T(3^{k+1}) = 3T(3^k) + 3^{k+1} \ .$$

We may now substitute for $T(3^k)$, using the inductive hypothesis. This will give us:

$$\begin{aligned} T(3^{k+1}) &= 3T(3^k) + 3^{k+1} \quad \text{by the form of the recurrence} \\ &= 3 \times 3^k k + 3^{k+1} \quad \text{by the inductive hypothesis} \\ &= 3^{k+1} k + 3^{k+1} \\ &= 3^{k+1}(k + 1) \end{aligned}$$

(b) Prove, by induction, that the recurrence $T(n) = 3T(n-1)+1$ with $T(0) = 0$ solves to $T(n) = (3^n - 1)/2$.

ANSWER:
Proof inductive on n.
Base Case: For $n = 0$, the expressions $T(0) = 0$ and $T(0) = (3^0 - 1)/2 = (1-1)/2 = 0$ both evaluate to 0.
Inductive Hypothesis: Assume that, for $n = k \geq 0$,it is indeed the case that

$$T(k) = \frac{3^k - 1}{2} \ .$$

Inductive Step: We want to show that, for $n = k+1$,

$$T(k+1) = \frac{3^{k+1-1}}{2} \ .$$

But, by the form of the recurrence, we have:

$$T(k+1) = 3T(k) + 1 \ .$$

We may now substitute for $T(k)$, using the inductive hypothesis. This will give us:

$$\begin{aligned} T(k+1) &= 3T(k)+1 \text{ by the form of the recurrence} \\ &= 3 \times \frac{(3^k - 1)}{2} + 1 \text{ by the inductive hypothesis} \\ &= \frac{3 \times 3^k - 3 + 2}{2} \\ &= \frac{3^{k+1} - 1}{2} \end{aligned}$$

Problem 4: (25 points)

(a) Suppose that have 2c coins and 5c coins. What amounts can you form using only such coins? (Give full proof/may need strong induction). Let x be an amount that can be formed using such coins. How can you combine 2c coins and 5c coins to form the number x, using the smallest possible number of coins? (Give full proof/may need strong induction).

ANSWER:
Claim 1: Every (integer) amount $n \geq 4$ can be formed using 2c and 5c coins.
Proof: Inductive on n. Strong induction.
Base Case: For $n = 4$ and $n = 5$, it is indeed true that $4 = 2 \times 2$ and $5 = 5 \times 1$.
Inductive Hypothesis:
Assume that, for some $k \geq 4$, every integer i where $4 \leq i \leq k$ can be written as

$$i = 2 \times p_i + 5 \times q_i \quad ,$$

where p_i and q_i are non-negative integers (that depend on i). Inductive Step: We want to show that, for $n = k + 1$,

$$k + 1 = 2 \times p_{k+1} + 5 \times q_{k+1} \quad ,$$

where p_{k+1} and q_{k+1} are non-negative integers (that depend on $k + 1$). But it is easy to see that:

$$\begin{aligned} k + 1 &= (k - 1) + 2 \\ &= 2 \times p_{k-1} + 5 \times q_{k+1} + 2 \;\; \text{by the inductive hypothesis} \\ &= 2 \times (p_{k-1} + 2) + 5 \times q_{k+1} \\ &= 2 \times p_{k+1} + 5 \times q_{k+1} \\ & \;\, \text{for } p_{k+1} = p_{k-1} + 2 \text{ and } q_{k+1} = q_k \quad . \end{aligned}$$

Remark: The above inductive proof suggests the following breakup of a number $n \geq 4$ to 2c and 5c coins:
If n is even, then use $n/2$ coins of 2c.
If n is odd, and since $n \geq 4$, then n can be written as $n = 5 + 2m = 5 + \frac{n-5}{2}$. The proof suggests to use one 5c coin, and $(n - 5)/2$ coins of 2c. Clearly this is not optimal. For example, the proof suggests to make 20c out of 10 coins of 2c. But it is clearly more efficient to use 4 coins of 5c.

We now have to proceed to find the optimal way of writing

$$n = 2 \times p_n + 5 \times q_n \quad ,$$

for each $n \geq 4$. By optimal, we mean that the total number of coins $p_n + q_n$ is as small as possible.

Using our ”strategic” solution building approach, we solve the problem optimally for a bunch of small values of n. Hopefully, a pattern will emerge.

n	p_n	q_n	$p_n + q_n$
4	2	0	2
5	0	1	1
6	3	0	3
7	1	1	2
8	4	0	4
9	2	1	3
10	0	2	2
11	3	1	4
12	1	2	3
13	4	1	5
14	2	2	4
15	0	3	3
16	3	2	5
17	1	3	4
18	4	2	6
19	2	3	5
20	0	4	4
21	3	3	6
22	1	4	5
23	4	3	7
24	2	4	6
25	0	5	5

Now the pattern speaks loudly in groups of 5. Moreover, the pattern suggests what we expected intuitively, nameley, that we should use as many 5c coins as possible, and make up the remainder using 2c coins, which will end up in a slightly different strong induction argument.

But before stating the claim in full generality, let us give one more table, making the pattern and its formalization explicit.

n	$n = 5 \times k + \alpha$	k	α	p_n	q_n	$p_n + q_n$
−−	−−−−−	−−	−−	−−	−−	−−−
4	$5 \times 0 + 4$	0	4	2	0	2
−−	−−−−−	−−	−−	−−	−−	−−−
5	$5 \times 1 + 0$	1	0	0	1	1
6	$5 \times 1 + 1$	1	1	3	0	3
7	$5 \times 1 + 2$	1	2	1	1	2
8	$5 \times 1 + 3$	1	3	4	0	4
9	$5 \times 1 + 4$	1	4	2	1	3
−−	−−−−−	−−	−−	−−	−−	−−−
10	$5 \times 2 + 0$	2	0	0	2	2
11	$5 \times 2 + 1$	2	1	3	1	4
12	$5 \times 2 + 2$	2	2	1	2	3
13	$5 \times 2 + 3$	2	3	4	1	5
14	$5 \times 2 + 5$	2	4	2	2	4
−−	−−−−−	−−	−−	−−	−−	−−−
15	$5 \times 3 + 0$	3	0	0	3	3
16	$5 \times 3 + 1$	3	1	3	2	5
17	$5 \times 3 + 2$	3	2	1	3	4
18	$5 \times 3 + 3$	3	3	4	2	6
19	$5 \times 3 + 4$	3	4	2	3	5
−−	−−−−−	−−	−−	−−	−−	−−−
20	$5 \times 4 + 0$	4	0	0	4	4
21	$5 \times 4 + 1$	4	1	3	3	6
22	$5 \times 4 + 2$	4	2	1	4	5
23	$5 \times 4 + 3$	4	3	4	3	7
24	$5 \times 4 + 4$	4	4	2	4	6
−−	−−−−−	−−	−−	−−	−−	−−−
25	$5 \times 5 + 0$	5	0	0	5	5

Now we can state the optimal solution formally: For every $n \geq 4$, if $n = 5k + \alpha$, with $0 \leq \alpha \leq 4$:
(a) If $\alpha = 0$ then the optimal solution consists of k coins of 5c.
(b) If $\alpha = 1$ then the optimal solution consists of $(k - 1)$ coins of 5c and 3 coins of 2c.
(c) If $\alpha = 2$ then the optimal solution consists of k coins of 5c and 1 coin of 2c.
(d) If $\alpha = 3$ then the optimal solution consists of $(k - 1)$ coins of 5c and 4 coins of 2c.
(e) If $\alpha = 4$ then the optimal solution consists of k coins of 5c and 2 coins of 2c.
In the next page we prove, by strong induction, that this is indeed the optimal solution.

Claim 2: Every number $n \geq 4$ can be written as $n = 2 \times p_n + 5 \times q_n$, where p_n and q_n are non-negative integers. If $n = 5k + \alpha$, for some $k \geq 0$, then $p_n + q_n$ is minimized by the following choices, and these are the *only* choices that minimize $p_n + q_n$:
(a) If $\alpha = 0$ then $q_n = k$ and $p_n = 0$.
(b) If $\alpha = 1$ then $q_n = (k - 1)$ and $p_n = 3$.
(c) If $\alpha = 2$ then $q_n = k$ and $p_n = 1$.
(d) If $\alpha = 3$ then $q_n = (k - 1)$ and $p_n = 4$.
(e) If $\alpha = 4$ then $q_n = k$ and $p_n = 2$.

Proof: Strong induction on n.
Base Case: The cases $n = 4, n = 5, n = 6, n = 7, n = 8, n = 9$ can be verified from the table in the previous page.
Inductive Hypothesis: Assume that $n = j \geq 10$, and for all i such that $4 \leq i \leq j$ the proposition of Claim 2 is true.
Inductive Step: We wish to show that the proposition of Claim 2 is true for $n = j + 1$.
We will prove this by contradiction. Assume, for the puproses of contradiction, that $j + 1 = 2p_{j+1} + 5q_{j+1}$, $p_{j+1} + q_{j+1}$ is minimized, and either p_{j+1} or q_{j+1} is not of the form indicated by cases (a) through (e).
Case 1: $q_{j+1} \geq 1$, that is, at least one 5c coin was used, and $j + 1 = 2p_{j+1} + 5q_{j+1}$ where $p_{j+1} + q_{j+1}$ was minimized for a choice of p_{j+1} and q_{j+1} other than the ones suggested in rules (a) through (e). We will argue that this cannot be true, by contradiction. In particular, it is easy to see that such a solution for $j + 1$ would imply a solution for $j' = (j + 1) - 5$ (by removing a 5c coin). different than the one suggested by rules (a) through (e). But, we know that for j' the inductive hypothesis applies, so such a different solution cannot exist.
Case 2: $q_{j+1} = 0$, that is, no 5c coins were used, and yet, $j + 1 = 2p_{j+1} + 5q_{j+1}$ and $p_{j+1} + q_{j+1}$ was minimized. Then, we can easily infer that $j + 1$ is an even number, and $p_{j+1} = (j + 1)/2$ is the optimal (using the smallest number of 2c and 5c coins) way of forming $j + 1$, namely by picking $(j + 1)/2$ coins of 2c each.
We will derive a contradition by showing that the combination of 5c and 2c coins suggested in cases (a) through (e) of the claim would have resulted in using less than $(j + 1)/2$ coins.
Case 2a: $j + 1 = 5k$, for some $k \geq 2$ (and k is even).
The assumption is that $j + 1 = 5k$ can be formed using $(j + 1)/2 = 5k/2$ coins, and that is optimal.
Therefore, using k coins, as suggested in (a) results in at least as large a number of coins.
Therefore:

$$\begin{array}{rcll} \frac{5k}{2} & \leq & k & \iff \\ 5k & \leq & 2k & \text{with } k \geq 2 \\ & & & \text{impossible/contradiction} \end{array}$$

Case 2b: $j + 1 = 5k + 1$, for some $k \geq 2$ (and k is odd).
The assumption is that $j + 1 = 5k + 1$ can be formed using $(j + 1)/2 = (5k + 1)/2$ coins, and that is optimal.
Therefore, using $(k - 1) + 3$ coins, as suggested in (b) results in at least as large a number of coins.
Therefore:

$$\begin{array}{rcll} \frac{5k+1}{2} & \leq & (k-1)+3 & \iff \\ 5k+1 & \leq & 2k+4 & \iff \\ 5k & \leq & 2k+3 & \text{with } k \geq 2 \\ & & & \text{impossible/contradiction} \end{array}$$

Case 2c: $j + 1 = 5k = 2$, for some $k \geq 2$ (and k is even).
The assumption is that $j + 1 = 5k + 2$ can be formed using $(j + 1)/2 = (5k + 2)/2$ coins, and that is optimal.
Therefore, using $k + 1$ coins, as suggested in (c) results in at least as large a number of coins.

Therefore:

$$\begin{array}{rcll} \frac{5k+2}{2} & \leq & k+1 & \iff \\ 5k+2 & \leq & 2k+2 & \iff \\ 5k & \leq & 2k & \text{with } k \geq 2 \\ & & & \text{impossible/contradiction} \end{array}$$

Case 2d: $j+1 = 5k+3$, for some $k \geq 2$ (and k is odd).
The assumption is that $j+1 = 5k+3$ can be formed using $(j+1)/2 = (5k+3)/2$ coins, and that is optimal.
Therefore, using $(k-1)+4$ coins, as suggested in (d) results in at least as large a number of coins.
Therefore:

$$\begin{array}{rcll} \frac{5k+3}{2} & \leq & (k-1)+4 & \iff \\ 5k+3 & \leq & 2k+6 & \iff \\ 5k & \leq & 2k+3 & \text{with } k \geq 2 \\ & & & \text{impossible/contradiction} \end{array}$$

Case 2e: $j+1 = 5k+4$, for some $k \geq 2$ (and k is even).
The assumption is that $j+1 = 5k+4$ can be formed using $(j+1)/2 = (5k+3)/2$ coins, and that is optimal.
Therefore, using $k+2$ coins, as suggested in (a) results in at least as large a number of coins.
Therefore:

$$\begin{array}{rcll} \frac{5k+4}{2} & \leq & k+2 & \iff \\ 5k+4 & \leq & 2k+4 & \iff \\ 5k & \leq & 2k & \text{with } k \geq 2 \\ & & & \text{impossible/contradiction} \end{array}$$

(b) Rosen, page 292, problem 14. (May need strong induction)

ANSWER:
Proof: Strong induction on n, and structural induction on the was n will repeatedly and recursively break down and build up the sum $S(n)$. Initially $S(n) = 0$. We may now think of the breaking down of n as creating a binary tree as follows:
The root is a node containing the number $n \geq 2$.
If $n = r + s$, with $r \geq 1$ and $s \geq 1$, then:
(a) We create two new nodes and connect them to the root as its left child and right child. The left child will contain the number r and the right child will contain the number s.
(b) We update the sum $S(n) := S(n) + r \times s$.
(c) If $r \geq 2$ then we recurse by going back to (a) and repeating the process thinking of r as n.
If $s \geq 2$ then we recurse by going back to (a) and repeating the process thinking of s as n.
The above process clearly builds a binary tree with n leaves. This is because eventually everything is broken down to units. (Of course, the binary tree that we built is neither complete nor balanced.)
Base Case: For $n = 1$, there is no way to break down 1, and $S(1) = 0$, which is indeed equal to $S(1) = (1 - 1) \times 1/2 = 0$.
Inductive Hypothesis: Assume that for some $k \geq 1$, and for all $1 \leq i \leq k$, it is indeed true that

$$S(i) = \frac{(i-1)i}{2} \quad .$$

Inductive Step: We want to show that

$$S(k+1) = \frac{k(k-1)}{2} \quad .$$

Now suppose that $k + 1 = r + s$, with $k \geq r \geq 1$ and $k \geq s \geq 1$, hence the inductive hypothesis applies to s and r. We therefore have:

$$\begin{aligned}
S(k+1) &= r \times s + S(r) + S(s) \\
&= r \times s + \frac{(r-1)r}{2} + \frac{(s-1)s}{2} \quad , \\
&\qquad \text{by the inductive hypothesis} \\
&= \frac{2rs + r^2 - r + s^2 - s}{2} \\
&= \frac{(r^2 + 2rs + s^2) - (r + s)}{2} \quad \text{, by regrouping} \\
&= \frac{(r+s)^2 - (r+s)}{2} \\
&= \frac{n^2 - n}{2} \quad \text{, by recalling that } n = r + s \\
&= \frac{(n-1)n}{2} \quad .
\end{aligned}$$