10-30-08 Homework 5
CS 1050 A Due: 11-6-08

Every assignment will be due at the beginning of class. Recall that you can collaborate in groups and/or use external references, but you must acknowledge the group/references that you used, and you must *always write your solutions alone.* Remember that for 90% of the people, more than 50% of the understanding happens during writing/implementation/etc. (And this is not true only for CS 1050. It is true for mostly everything, at least technical).

Please read the entire homework before starting to work on it. This homework is a survey of basic proofs techniques (direct and indirect), and induction. It also touches on ideas of growth of functions, and elementary probability (that are our next topics).

Please stop by for questions during office hours of instructor or TAs and send email to mihail@cc.gatech.edu with title 1050 at any time. This helps you, but it also helps us! Sometimes it helps us understand where the class stands and where we should put more or less emphasis. And sometimes, you give us presentational and technical ideas that we would have not thought of otherwise. So keep all communication links open!

Please print this document, and write your solutions on the printout.
Please hand-in the completed printout.

PRINT YOUR NAME HERE:... ....

WRITE YOUR EMAIL HERE:... ....

Problem 1: (25 points)

One morning, early at sunrise, a Buddist monk began to climb a tall mountain. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit.

The monk ascended the path at varying rates of speed, stopping many times along the way to rest and to eat the dried fruit that he carried with him. He reached the temple shortly before sunset. After several days of fasting and meditation he began his journey back along the same path, starting at sunrise, and again walking at variable speeds with many pauses along the way. His average speed descending was, of course, greater than his average climbing speed.

Prove that there is a spot along the path that the monk will occupy on both trips at precisely the same time of day.

Hint: The problem has a one line direct proof.

Problem 2: (25 points)

Let $x_1, x_2, \ldots x_n$ be non-negative numbers (and assume without loss of generality that n is even). Consider their average

$$\mu = \frac{x_1 + x_2 + \ldots x_n}{n} \quad .$$

(a) Prove that none of the x_i's can be greater than $n \times \mu$. Hint: Realize that this is a very natural statement. How could the average of n non-negative numbers be μ, if just one single x_i was larger than $n \times \mu$? Even if all the other x_i's were zero, the single huge x_i would bring the average higher than μ... So proceed with a proof by contrapositive or contradiction.

(b) Prove that no more than $n/2$ of the x_i's can be greater than 2μ. Hint: Realize that this is also a very natural statement. How could the average of n non-negative numbers be μ, if more than half of the x_i's was larger than 2μ? Even if all the remaining x_i's were zero, the large x_i's would bring the average higher than μ... So proceed with a proof by contrapositive or contradiction.

(c) Construct a few examples of the statement that you proved in Problem 2, part (b). Pick $n = 6$ non-negative numbers, and think of them as representing monthly salaries of 6 people. Let the salaries range sunstantially. Some quite big, and some quite small. Then compute the average salary of the group. Verify that no more than 3 people make more than twice the average salary of the group.
Repeat 2-3 times with different numbers. Verify that, each time, no more than half the people make more than twice the average salary.

Problem 3: (25 points)

(a) Consider the recurrence $f(n) = f(n-1) + 10{,}000$, with $f(1) = 10{,}000$. Prove that the solution to this recurrence is $f(n) = 10{,}000 \times n$.

(b) Consider the recurrence $F(n) = 10F(n-1)$, with $F(1) = 10$. Prove that the solution to this recurence is $F(n) = 10^n$.

(c1) Show that, for $n = 1$, $n = 2$, $n = 3$ and $n = 4$, $F(n) < f(n)$.
(c2) Prove, by induction on n, that, for every $n \geq 5$, $f(n) < F(n)$.

Problem 4: (25 points)

Suppose that have 2c coins and 3c coins. What amounts can you form using only such coins? Let x be an amount that can be formed using such coins. How can you combine 2c coins and 3c coins to form the number x, using the smallest possible number of coins? Hint: Read solutions to Homework 4, where we analyzed a very similar problem.

Extra Credit: (20 points)

(a) Let $x = x_n x_{n-1} \dots x_1 x_0$ be an $(n+1)$-digit integer number in decimal representation. Prove that x is a multiple of 11 if and only if $x_n - x_{n-1} + x_{n-2} - x_{n-3} + \dots + (-1)^{n-1} x_1 + (-1)^n x_0$ is a multiple of 11.

Extra Credit: (20 points)

An unusual parlor trick is performed as follows. Ask a spectator A to write down any three-digit number, and then to repeat the digits in the same order to make a six digit number (e.g. 394,394). With your back turned so that you cannot see the number, ask A to pass the sheet of paper to spectator B, who is requested to divide the number by 7.

"Don't worry about the remainder", you tell him, "because there will not be any". B is surprised to discover that you are right (e.g. 394,394 divided by 7 is 56,342). Without telling you the result, he passes it on to spectator C, who is told to divide it by 11. Once again you state that there will be no remainder, and this also proves correct (56,342 divided by 11 is 5,122).

With your back still turned, and no knowledge whatever of the figures obtained by these computations, you direct a fourth spectator, D, to divide the last number by 13. Again the division comes out even (5,122 divided by 13 is 394). This final result is written on a slip of paper which is folded and handed to you. Without opening it, you pass it on to spectator A.

"Open this," you tell him, "and you will find your original three-digit number."

Prove that the trick will not fail to work, regardless of the digits chosen by the first spectator.