**Problem 1: Greedy Vertex Cover**

Perhaps the first strategy one tries when designing an algorithm for an optimization problem is the greedy strategy. For the unweighted vertex cover problem, this would involve iteratively picking a maximum degree vertex and removing it, together with edges incident at it, until there are no edges left.

(a) Show that this algorithm achieves an approximation guarantee of $O(\log n)$.

(b) Give a tight example: Class of input instances where this algorithm performs as bad as $\Omega(\log n)$.

**Problem 2: Set Coverage**

Maximum set coverage is the following problem: Given a set $U$ of $n$ elements, a collection of subsets of $U$, $S_1, \ldots, S_m$, and an integer $k$, pick $k$ sets so as to maximize the number of covered elements.

(a) Show that maximum set coverage is NP-hard.

(b) Show that the obvious algorithm, of greedily picking the best set in each iteration until $k$ sets are picked, achieves an approximation factor of $1 - \left(1 - \frac{1}{k}\right)^k > 1 - \frac{1}{e}$.

**Problem 3: MAX-SAT**

Recall that MAX-SAT is the following problem: Given a conjunctive normal form formula $f$ on Boolean variables $x_1, \ldots, x_n$, and non-negative weights, $w_c$, for each clause $c$ of $f$, find a truth assignment to the Boolean variables that maximizes the total weight of satisfied clauses.

(a) Show that the following is a factor $1/2$ approximation algorithm for MAX-SAT. Let $\tau$ be an arbitrary truth assignment, and $\tau'$ be its complement, i.e., a variable is True in $\tau$ if and only if it is False in $\tau'$. Compute the weight of clauses satisfied by $\tau$ and $\tau'$, then output the better assignment.

(b) Give a tight example: Class of input instances where this algorithm performs as bad as $1/2$.

**Problem 4: Fractional MAX-CUT**

Let $G(V, E)$ be an unweighted, undirected graph, $|V| = n$. Recall that MAX-CUT (which is well known to be NP-hard) can be formalized as:

$$\text{maximize} \quad \sum_{\{u,v\} \in E} |x_u - x_v|$$

subject to:

- $x_u \in \{0, 1\}$
- $0 \leq u \leq n$

Show that MAX-CUT remains NP-hard, even if we relax the integrality constraint. In particular, show that the following problem is NP-hard:

$$\text{maximize} \quad \sum_{\{u,v\} \in E} |x_u - x_v|$$

subject to:

- $0 \leq x_u \leq 1$
- $0 \leq u \leq n$