Problem 1: Primal-Dual, Exact Complementary Slackness

This exercise is a review of the basics of LP-duality (chapter 12 in Vazirani’s book).

(a) Let $G(V, E)$ be an undirected graph with source $s \in V$ and sink $t \in V \setminus \{s\}$, and edge capacities $c : E \to R^+$. Replace each undirected edge $\{i, j\}$ with two directed edges $(i, j)$ and $(j, i)$ of equal capacities $c_{ij} = c_{ji}$. Introduce a link of infinite capacity from $t$ to $s$, and a link of zero capacity from $s$ to $t$: $c_{ts} = \infty$ and $c_{st} = 0$. Then the straightforward formalization of max $s$-$t$ flow in $G$ is:

\[
\begin{align*}
\text{maximize} & \quad f_{ts} \\
\text{subject to} & \quad f_{ij} \leq c_{ij} \quad \forall (i, j) \in E \\
& \quad \sum_{j: (j,i) \in E} f_{ji} - \sum_{j: (i,j) \in E} f_{ij} = 0 \quad \forall i \in V \\
& \quad f_{ij} \geq 0 \quad \forall (i, j) \in E
\end{align*}
\]

In class we formalized max $s$-$t$ flow with the slightly different LP (2) below, where the set of flow preservation constraints of (1) are replaced with a set of corresponding constraints that may initially seem weaker.

\[
\begin{align*}
\text{maximize} & \quad f_{ts} \\
\text{subject to} & \quad f_{ij} \leq c_{ij} \quad \forall (i, j) \in E \\
& \quad \sum_{j: (j,i) \in E} f_{ji} - \sum_{j: (i,j) \in E} f_{ij} \leq 0 \quad \forall i \in V \\
& \quad f_{ij} \geq 0 \quad \forall (i, j) \in E
\end{align*}
\]

Prove that the constraints in (2): $\sum_{j: (j,i) \in E} f_{ji} - \sum_{j: (i,j) \in E} f_{ij} \leq 0, \forall i \in V$, imply the flow preservation constraints in (1): $\sum_{j: (j,i) \in E} f_{ji} - \sum_{j: (i,j) \in E} f_{ij} = 0, \forall i \in V$, thus showing that (1) and (2) are equivalent.

(b) The formalization of max $s$-$t$ flow given in class, namely the LP (2) above, has two important features: (i) Its dual (DP) gives fundamental intuition on fractional cuts. (ii) Its dual (DP) is also an ”exact relaxation” of the min $s$-$t$ cut problem, in the sense that (DP) has an integral optimal solution which corresponds to a min $s$-$t$ cut in the natural way. The dual of (2) is:

\[
\begin{align*}
\text{minimize} & \quad \sum_{(i,j) \in E} c_{ij} d_{ij} \\
\text{subject to} & \quad d_{ij} - p_i + p_j \geq 0 \quad \forall (i, j) \in E \\
& \quad p_s - p_t \geq 1 \\
& \quad d_{ij} \geq 0 \quad \forall (i, j) \in E \\
& \quad p_i \geq 0 \quad \forall i \in V
\end{align*}
\]

In the rest of this exercise you will be given an instance of maxflow, and three flow augmentation paths. You will then be asked to show how the particular flow augmentation paths, in combination with (2) and (3), establish total flow is optimality, and a corresponding mincut.

The maxflow instance is $V = \{s, 1, 2, 3, t\}$, $E = \{\{s, 1\}, \{1, 2\}, \{s, 2\}, \{s, 3\}, \{1, t\}, \{2, t\}, \{3, t\}\}$, and capacities $c_{s1} = c_{1s} = 5$, $c_{12} = c_{21} = 6$, $c_{s2} = c_{2s} = 4$, $c_{s3} = c_{3s} = 1$, $c_{1t} = c_{t1} = 3$, $c_{2t} = c_{t2} = 2$, $c_{3t} = c_{t3} = 7$. The three flow augmentation paths are: $(s, 2, 1, t)$ with flow 3, followed by $(s, 1, 2, t)$ with flow 2, followed by $(s, 3, t)$ with flow 1.

- Write (2) and (3) for the above maxflow instance, including all the constraints explicitly.
- Initialize with (2) feasible by setting $f_{ij} = 0$ for all $(i, j) \in E$, and (3) infeasible by setting $p_s = 1$, $p_t = -1$, $d_{ij} = 0$ for all $(i, j) \in E$. 

\( p_i = 0 \) for all \( i \in V \setminus \{s\} \), and \( d_{ij} = 0 \) for all \( (i, j) \in E \).

- Show how the three augmentation paths can be used to update, in three rounds, the variables of (2) and (3), so that:
  - (2) remains always feasible, while the objective function \( f_{ts} \) increases in every round.
  - The variables of (3) are updated integrally by restricting the updates to values \( d_{ij} \in \{0, 1\} \), \( \forall (i, j) \in E \), and \( p_i \in \{0, 1\} \), \( \forall i \in V \).
  - At the end of the third round (3) is feasible (number of violated constraints decreases).
  - At the end of the third round, all complementary slackness conditions hold with equality - thus you know you have an optimal solutions for both (3) and (2).

- Indicate the integral mincut suggested by the values of (3) at the end of the third round.

**Problem 2: Primal-Dual, Relaxed Complementary Slackness**

Consider the weighted vertex cover Algorithm 2.17, in page 24 of Vazirani’s book. Show that this is a factor 2 approximation algorithm by arguing (a) that it is a primal-dual algorithm (you have to write the primal and the dual), and (b) the solution ensures that, in the end, relaxed primal and dual complementary slackness conditions hold and ensure factor 2 performance guarantee. (For exact and relaxed slackness, review pages 97 and 125 of Vazirani’s book, and the obvious context surrounding them).

**Problem 3: Set Cover Factor \( f \) in Parallel, \( f \) is frequency of most frequent element**

Let \( \Omega \) be a collection of elements. Let \( S \) be collection of subsets of \( \Omega \), where each set \( S \in S \) has an associated positive cost \( c_S \). Consider the following set cover algorithm:

\[
\begin{align*}
    k &:= 0; \\
    \text{initially, all elements are uncovered and the cover is the emptyset: } U := \Omega; C := \emptyset; \\
    \text{while } U \neq \emptyset, \text{ ie there are uncovered elements} \\
    k &:= k + 1; \\
    \forall e \in U, \text{ set currentcost}(e) = \min_{S: |S \cap U| > 0} \frac{c_S}{|S \cap U|}; \\
    \forall S : |S \cap U| > 0, \text{ if } \sum_{e \in S} \text{currentcost}(e) \geq \frac{ck}{2} \text{ then} \\
    C &:= C \cup \{S\}; \\
    U &:= U \setminus S;
\end{align*}
\]

(a) Argue that the above algorithm achieves approximation factor \( 3f \), where \( f \) is the frequency of the most frequent element: \( f = \max_{e \in \Omega} |\{S \in S : e \in S\}| \).

(b) Argue that at the end of the execution, \( k \leq \left( \log n + \log \frac{\max_{S \in S} c_S}{\min_{S \in S} c_S} \right) \).

**Problem 4: Half Integral Vertex Cover, Efficient Combinatorial Algorithm**


**Problem 5: Extra Credit**

Work on either problem 12.10 (MST via exact LP-relaxation), or problem 12.11 (von Newmann’s minimax theorem in game theory derived from the LP-duality theorem), or both!

All thoughts welcome!