For this midterm exam you have to work by yourselves, and write solutions by yourselves. You may use any outside references you want, as long as you acknowledge them in your solutions. The instructor mihail@cc.gatech.edu is always available to answer questions, and so is the TA Varum Kanade varunk@cc.gatech.edu.

Problem 1

(a) Consider the Steiner tree factor 2 approximation algorithm that is developped in Sectio 3.1 (pages 27-29) of Vazirani's book. What is the performance of this algorithm, if we run it on an instance of MST (ie, there are no Steiner vertices)? Explain your answer.
(b) Consider the primal-sual Steiner Forest factor 2 approximation algorithms that is developed in class and written in detail in Chapter 22 (pages 195-206) of Vazirani's book. What is the performance of this algorithm, if we run it on an instance of MST (ie, all vertices have to be connected, and there are no Steiner vertices)? Explain your answer.

Problem 2

Consider the facility location and k-medial approximation algorithms developed in class (with approximation factors 3 and 6 respectively); these algorithms and their analysis are written in full detail in Chapters 24 and 25 of Vazirani's book. These approxination factor guarantees apply when costs of links follow triangular inequality.
(a) Identify exaclty the places of the analysis where triangular inequality is used.
(b) Would you be able to get a performance guarantee if the usual triangular inequality $\text{cost}(uv) \leq \text{cost}(uw)+\text{cost}(wv)$ was relaxed to $\text{cost}(uv) \leq 2(\text{cost}(uw)+\text{cost}(wv))$? Explain your answer.

Problem 3

Consider the myltiway cut problem randomized approximation algorithm that was developed in class, with approximation factor $1.5(1-1/k)$, in expectation; this algorithm and its analysis is written in full detail in Paragraphs 19.1 and 19.2 (pages 154-159) of Vazirani's book.
(a) Consider Algorithm 19.4 is page 157. How would the approximation guarantee change if σ was picked to be a random permutation form S_k (the set of all permutations over k elements)?
(b) Obtain a randomized approximation algorithm for the multiway cut problem which outputs a solution of cost no more that a 1.5 factor from OPT, with high probability. Hint: Show that Lemma 19.6 in page 158 implies that $\Pr[\text{cost}(C) \leq 1.5\text{OPT}] \geq 2/k \geq 2/n$. Then run Algorithm 19.4 polynomially many times, and output the best cut.

Problem 4

We have partly discussed the two questions below in class, so you may want to look at the class notes. You may also find guidance in Chapter 13 of Vazirani's book.
(a) The set multicover problem is as follows: There are n elements. Each element e needs to be covered a specified integer number, r_e, of times. Each set can be included in the cover more than once. The objective gain is to cover all elements at least up to their coverage requirements at minimum cost. We will assume that the cost of picking a set S k times is $k\text{cost}(S)$. Give an H_n factor approximation algorithm for the set multicover problem.
(b) The mutliset multicover problem is as follows: There are n elements. A multiset contains a specified number of copies of each element and it can be picked several times. Let $M(S,e)$ denote the multiplicity of element e in set S. The instance satisfies the condition that the multiplicity of an element in each set is at most its coverage requirement, i.e., $\forall S, e$ $M(S,e) \leq r_e$ The objective

again is to cover all elements at least up to their coverage requirement at minimum cost. Give an H_m factor approximation algorithms for the multiset multicover problem, where m is the size of the largest multiset in the given instance (the size of amultiset counts elements with multiplicity).

Problem 5, Extra Credit

Let $G(V, E)$ be an undirected unweighted graph. In class we saw that the minimum vertex cover problem is NP-complete, and we gave a factor 2 (efficient) approximation algorithm for minimum vertex cover. An independent set of $G(V, E)$ is a subset of vertices $I \subseteq V$ such that there is no edge in E with both its endpoints in I: $\forall \{u, v\} \in E$, $|\{u, v\} \cap I| \leq 1$. The maximum independent set problem asks for an independent set of maximum cardinality.

(a) Argue that, if there is a polynomial time algorithm to solve maximum independent set, then there is a polynomial time algorithm to solve minimum vertex cover, thus showing that maximum independent set is NP-complete.

(b) Argue that, the factor 2 (efficient) approximation algorithm for minimum vertex cover is not immediately useful to get a constant factor (efficient) approximation algorithm for maximum independent set. That is, show that the reduction of part (a) does not preserve approximations.

(c) Give an efficient approximation algorithm for maximum independent set where, for a graph $G(V, E)$, the performance guarantee is $O(2|E|/|V|)$ (the average degree of G). That is, of I is the output of your algorithm and I^* is a maximum independent set, then $|I^*| = O(2|I||E|/|V|)$.