10-28-08
CS 1050 A

Quiz 2

Time: 80 mins

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Problem 1: 25 points (Direct Proofs)

(a) Prove that, for every real number x, the quantity $x^4 - x^2 + 2x + 4000$ is always positive. State clearly the proof technique that you are using. If you use a direct proof, state clearly the general principle(s) that you are using.

ANSWER:
Direct Proof.
General Principles: The square of any real number is always non-negative. The sum of non-negative quantities and at least one positive quantity is always positive.

$$\begin{aligned} x^4 - x^2 + 2x + 4000 &= x^4 - 2x^2 + x^2 + 2x + 1 + 1 + 3998 \\ &= (x^4 - 2x^2 + 1) + (x^2 + 2x + 1) + 3998 \\ &= (x^2 - 1)^2 + (x + 1)^2 + 3998 \\ &> 0 \ . \end{aligned}$$

(b) Prove that, for every integer n, the product $n(n+1)(n+2)(n+3)$ is always divisible by 8. State clearly the proof technique that you are using. If you use a direct proof, state clearly the general principle(s) that you are using.

ANSWER:
Direct Proof.
General Principles: Exactly one of every four consecutive numbers is a multiple of 4. If x is an even number, then $x+2$ and $x-2$ are even numbers.

Let $\prod = n(n+1)(n+2)(n+3)$. We know that exactly one of n, $n+1$, $n+2$ and $n+3$ is a multiple of 4.
<u>Case 1:</u> $n = 4k$, and hence $n+1 = 2l$, and hence $\prod = 4k(n+1)2l(n+3) = 8kl(n+1)(n+3)$, obviously a multiple of 8.
<u>Case 2:</u> $n+1 = 4k$, and hence $n+3 = 2l$, and hence $\prod = n4k(n+2)2l = 8kln(n+2)$, obviously a multiple of 8.
<u>Case 3:</u> $n+2 = 4k$, and hence $n = 2l$, and hence $\prod = 2l(n+1)4k(n+3) = 8kl(n+1)(n+3)$, obviously a multiple of 8.
<u>Case 4:</u> $n+3 = 4k$, and hence $n+1 = 2l$, and hence $\prod = n2l(n+2)4k = 8kln(n+2)$, obviously a multiple of 8.

Problem 2: 25 points (Indirect Proofs)

(a) Prove that, for every non-negative integer n, $\sqrt{8n+5}$ is an irrational number. State clearly the proof technique that you are using.

ANSWER: Proof by contradiction.
Assume, for the purposes of contradiction, that for some integer n, $\sqrt{8n+5}$ is a rational number.
Therefore, there exist p and q relatively prime, such that $\sqrt{8n+5} = p/q$ which implies

$$(8n+5)q^2 = p^2 \quad (*).$$

What are p and q? Note that $8n+5$ is always odd. Therefore, if either p or q was even, then that would force one side of (*) to be even, so the other side of (*) would have to be even, this forcing both p and q to be even, which contradicts the fact that p and q are relatively prime.
Therefore, the only possibility is that both p and q are odd.
So let $p = 2k+1$, for some integer k, and let $q = 2l+1$, for some integer l, and let us substitute these values to (*):

$$\begin{aligned}
(8n+5)(2k+1)^2 &= (2l+1)^2 && \Longleftrightarrow \\
(8n+5)(4k^2+4k+1) &= (4l^2+4l+1) && \Longleftrightarrow \\
32nk^2+32nk+8n+20k^2+20k+5 &= 4l^2+4l+1 && \Longleftrightarrow \\
32nk^2+32nk+8n+20k^2+20k+4 &= 4l^2+4l && \Longleftrightarrow \\
8nk^2+8nk+2n+5k^2+5k+1 &= l^2+l && \Longleftrightarrow \\
8nk^2+8nk+2n+5k(k+1)+1 &= l(l+1)
\end{aligned}$$

But the left hand side above is ODD: $8nk^2+8nk+2n$is even, $k(k+1)$ is the product of two consecutive integers and hence it is even, and we add 1.
On the other hand, the right hand side above is EVEN (product of two consecutive integers).
So we get ODD=EVEN, which is a contradiction.

(b) Let $x_1 \geq 0$ and $x_2 \geq 0$ be real numbers. Let μ be their average, namely

$$\mu = \frac{x_1 + x_2}{2}.$$

Prove that either $x_1 \leq \mu$ or $x_2 \leq \mu$. State clearly the proof technique that you are using.

ANSWER: Proof by contrapositive.
For real numbers $x_1 \geq 0$ and $x_2 \geq 0$, we have to show:

$$\left(\mu = \frac{x_1 + x_2}{2}\right) \Longrightarrow (x_1 \leq \mu \vee x_2 \leq \mu) \quad .$$

We will equivalently show

$$\neg (x_1 \leq \mu \vee x_2 \leq \mu) \Longrightarrow \neg \left(\mu = \frac{x_1 + x_2}{2}\right) \quad .$$

which traslates to

$$(x_1 > \mu \wedge x_2 > \mu) \Longrightarrow \left(\mu \neq \frac{x_1 + x_2}{2}\right) \quad .$$

But the latter statement is obvious:

$$\begin{array}{rcll} x_1 & > & \mu & \\ x_2 & > & \mu & \\ \text{add} & \text{both} & \text{sides} & \Longrightarrow \\ (x_1 + x_2) & > & 2\mu & \Longrightarrow \\ \frac{x_1+x_2}{2} & > & \mu & \Longrightarrow \\ \frac{x_1+x_2}{2} & \neq & \mu & \end{array}$$

Problem 3: 25 points (Simple Induction)

(a) Consider the recurrence $f(n) = f(n-1)+2$, with $f(0) = 10$. Prove that $f(n) = 2n+10$. State clearly the proof technique that you are using. If you use induction, state clearly the base case, inductive hypothesis, and inductive step.

ANSWER: Proof by simple induction on n.
Base Case: For $n = 0$, we can verify that the expressions $f(0) = 10$ and $f(0) = 2\times0+10 = 10$ evaluate to the same number.
Inductive Hypothesis: For $n = k \geq 0$, assume that it is true that

$$f(k) = 2k + 10 \quad .$$

Inductive Hypothesis: For $n = k \geq 0$, we want to show that

$$f(k+1) = 2(k+1) + 10 \quad .$$

As always, our goal is to *bring in explicitly the expression involved in the inductive hypothesis.* This is done as follows:

$$\begin{aligned} f(k+1) &= f(k) + 10 \quad \text{by the definition of the recurrence} \\ &= (2k+2) + 10 \quad \text{by the inductive hypothesis} \\ &= 2(k+1) + 10 \quad . \end{aligned}$$

(b) Consider the recurrence $F(n) = 4F(n-1)+2$, with $F(0) = 0$. Prove that $F(n) = 2\frac{4^n-1}{3}$. State clearly the proof technique that you are using. If you use induction, state clearly the base case, inductive hypothesis, and inductive step.

ANSWER: Proof by simple induction on n.
Base Case: For $n = 0$, we can verify that the expressions $F(0) = 0$ and $F(0) = 2\frac{4^0-1}{3} = 2\frac{1-1}{3} = 0$ evaluate to the same number.
Inductive Hypothesis: For $n = k \geq 0$, assume that it is true that

$$F(k) = 2\frac{4^k - 1}{3} \quad .$$

Inductive Hypothesis: For $n = k \geq 0$, we want to show that

$$F(k+1) = 2\frac{4^{k+1} - 1}{3} \quad .$$

As always, our goal is to *bring in explicitly the expression involved in the inductive hypothesis.* This is done as follows:

$$\begin{aligned}
F(k+1) &= 4F(k) + 2 \quad \text{by the definition of the recurrence} \\
&= 4 \times 2 \times \frac{4^k - 1}{3} + 2 \quad \text{by the inductive hypothesis} \\
&= 2\frac{4 \times 4^k - 4}{3} + \frac{6}{3} \\
&= 2\frac{4 \times 4^k - 4}{3} + 2 \times \frac{3}{3} \\
&= 2 \times \frac{4^{k+1} - 4 + 3}{2} \\
&= 2 \times \frac{4^{k+1} - 1}{2} \quad .
\end{aligned}$$

(c) Consider the functions $f(x) = 2x + 10$ and $F(x) = 2^{\frac{4^x - 1}{3}}$ that were the solutions of the recurrences of parts (a) and (b) of this problem, for all real numbers $x \geq 0$. Compute $f(x)$ and $F(x)$, for $x = 0$, $x = 1$, $x = 2$, $x = 3$ and $x = 4$. Do a rough draw of the plots of $f(x)$ and $F(x)$, for all real numbers $x \geq 0$.

Problem 4: 25 points (Structural Induction, Strong Induction)

(a) Define a *complete binary tree* on 2^n leaves and *root* r, $n \geq 0$, recursively as follows:
For $n = 0$, the complete binary tree on $2^0 = 1$ leaf is a single node. We also call this node the "root" (that is, for $n = 0$, there is a single leaf and a single root which are the same node).
For $n > 0$, the complete binary tree on 2^n leaves consists of:
(I) A left subtree which is a complete binary tree on 2^{n-1} leaves and root r_{left}.
(II) A right subtree which is a complete binary tree on 2^{n-1} leaves and root r_{right}.
(III) An additional node r, called the "root", connected to r_{left} and r_{right}.

Define the *depth* of a complete binary tree on 2^n nodes, $n \geq 0$, as the distance (length of a path) from the root to a leaf. Prove that, for all $n \geq 0$, the depth of a complete binary tree on 2^n leaves is n. State clearly your proof technique. If you use induction, state clearly the base case, inductive hypothesis, and inductive step.

ANSWER: Simple structural induction on n.
Base Case: For $n = 0$, complete binary tree on $2^0 = 1$ node has depth 0 (the sole node is both the root and the sole leaf, so it has distance 0 to itself).
Inductive Hypothesis: Assume that for $n = k \geq 0$, it is indeed true that the depth of a complete binary tree on 2^k leaves is k.
Inductive Step: We want to argue that the depth of a complete binary tree on 2^{k+1} leaves is $k + 1$.
But, by construction, a complete binary tree on 2^{k+1} leaves consists of a left subtree and a right subtree, whose roots are connected to the new global root. But each of the left and right subtrees is a complete binary tree on 2^k leaves and hence of depth k, by the inductive hypothesis. So the depth of the entire tree is $k + 1$: 1 step to go from the global root to the root of either the left or the right subtree, and then k more steps to reach a leaf, by the inductive hypothesis.

(b) Show that, using 4c and 6c coins, you can form any even integer number of $2n$ cents, $n \geq 2$. State clearly what proof technique you are using. If you use an inductive proof, state clearly the base case, inductive hypothesis, and inductive step.

ANSWER: Strong induction on n.
Base Case: For $n = 2$ we have $2 \times 2 = 4 \times 1$, and for $n = 3$ we have $2 \times 3 = 6 \times 1$.
Inductive Hypothesis: Assume that for $n = 2k \geq 4$, and for every i such that $2 \leq i \leq k$, it is indeed true that

$$2i = 4p_i + 6q_i \quad ,$$

for non-negative integers p_i and q_i.
Inductive Step: We want to show that, for $n = k + 1$, there exist non-negative integers p_k and q_k, such that

$$2(k+1) = 4p_k + 6q_k \quad .$$

But, we can write:

$$2(k+1) = 2(k-1) + 4$$

and apply the inductive hypothesis to $(k-1)$:

$$2(k-1) = 4p_{k-1} + 6q_{k-1} \quad .$$

We therefore get:

$$\begin{aligned} 2(k+1) &= 2(k-1) + 4 \\ &= 4p_{k-1} + 6q_{k-1} + 4 \\ &= 4 \times (p_{k-1} + 1) + 6q_{k-1} \quad . \end{aligned}$$

So we have identified $p_{k+1} = p_{k-1} + 1$ and $q_{k+1} = q_{k-1}$.