ORDER OF GROWTH BASICS

You should be able to identify the leading term of a function:

Examples:

\[ f(x) = x^2 + 100 \log x + 3^x + \sqrt{x} \]

This is the leading term

\[ f(x) = x^2 + 100 \log x + \sqrt{x} \]

This is the leading term

\[ f(x) = \sqrt{x} \log^2 x + x \]

This is the leading term

\[ f(x) = 3^x + 3^x \times 100 \]

This is the leading term
Very broadly speaking:

1. **Exponential Functions** (grow faster than all)
2. **Polynomial Functions**
3. **Logarithmic Functions** (grow faster than all)
4. **Constant**

Symbolic representation:
- $f(x)$
- $\log x$, $\log x$, $\log x$, $\log x$
- $x$, $x^2$, $x^3$, $x^{1/2}$
- $3x$, $2x$, $x$, $x^{1/100}$

Relationships:
- $x$ (constant)
- $\log x$ (logarithmic)
- Exponential functions grow faster than all other functions.
Big O Notation

\[ f(x) = \mathcal{O}(g(x)) \]

means

\( f(x) \) grows at most as fast as \( g(x) \)

up to multiplicative constants,

ie we put \( x, 10x, 100x, 1,000,000x \) in the same category
**Definition** Let \( f(x), g(x) \) be non-negative functions for \( x \geq 0 \).

Say \( f(x) = O(g(x)) \) if and only if

\[
\exists C > 0 \quad \exists x_0.
\]

- \( f(x) \leq C g(x) \)
- \( x \geq x_0 \)

After \( x_0 \), \( f(x) \) is never greater than \( C g(x) \).
Example:
Show that \( 100x^4 + 10 = O(x^5) \)

Proof:
Let \( f(x) = 100x^4 + 10 \) and \( g(x) = x^5 \)

Using the definition of the previous page, we need to find a specific positive constant \( C \), and a specific \( x_0 \), such that \( f(x) \leq Cg(x) \), whenever \( x \geq x_0 \).

\[
\begin{align*}
\text{build up } f(x) & \quad \text{take multiples of } g(x) \\
100x^4 & \leq 100x^5, \text{ whenever } x \geq 1 \\
10 & \leq x^5, \text{ whenever } x \geq 2 \\
\hline
100x^4 + 10 & \leq 101x^5,
\end{align*}
\]
A VERY EASY WAY TO PROVE $f(x) = O(g(x))$

**Theorem:** If 
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = C,
\]
for some constant $C \geq 0$

then 
\[
f(x) = O\left(g(x)\right)
\]

**Proof:** 
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = C \Rightarrow \lim_{x \to \infty} \frac{f(x) - C}{g(x)} = 0
\]

By basic calculus, this means:
\[
\forall \varepsilon > 0 \exists x_0 : \frac{f(x) - C}{g(x)} < \varepsilon, \quad \forall x \geq x_0.
\]

Pick any $\varepsilon > 0$ and fix it. We now have:
\[
\frac{f(x) - C}{g(x)} < \varepsilon, \quad \forall x \geq x_0 \Rightarrow
\]
\[
\frac{f(x)}{g(x)} < \varepsilon + C, \quad \forall x \geq x_0 \Rightarrow
\]
\[
f(x) < (\varepsilon + C)g(x), \quad \forall x \geq x_0.
\]
Now let $c^* = (\varepsilon_0 + c)$

We have shown that

$\exists c^* > 0 \exists x_0 : f(x) < c^* g(x),$

$\forall x \geq x_0,$

$c^*: = \varepsilon_0 + c$

Thus satisfying the definition in page A4,

therefore, $f(x) = O( g(x) )$
Example Prove that \( \log_e x = O(x) \).

Proof: We will use the theorem saying
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = c \implies f(x) = O(g(x)).
\]

Let \( f(x) = \log_e x \), \( g(x) = x \).

\[
\lim_{x \to \infty} \frac{\log_e x}{x} = \text{de l' Hospital}
\]

\[
\lim_{x \to \infty} \frac{(\log_e x)'}{(x)'} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = 0
\]

Therefore, \( \log_e x = O(x) \).
Example

Prove that \( \log_e^3 x = O\left(\sqrt{x}\right) \).

Proof:

We will use the theorem saying

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = c \Rightarrow f(x) = O\left(g(x)\right).
\]

Let \( f(x) = \log_e^3 x \) and let \( g(x) = \sqrt{x} = x^{\frac{1}{2}} \).

\[
\lim_{x \to \infty} \frac{\log_e^3 x}{x^{\frac{1}{2}}} = \text{ de l'Hospital }
\]

\[
\lim_{x \to \infty} \left(\frac{\log_e^3 x}{x^{\frac{1}{2}}}\right)' =
\]

\[
\lim_{x \to \infty} \frac{3\left(\log_e^2 x\right) \cdot \frac{1}{x}}{\frac{1}{2} x^{-\frac{1}{2}}}
\]

\[
\lim_{x \to \infty} \frac{2 \times 3 \left(\log_e^2 x\right)}{x^{\frac{1}{2}}} = \text{ de l' Hospital }
\]
\[
\lim_{x \to \infty} 6 \left( \frac{2 \log \varepsilon x}{x^{\frac{1}{2}}} \right) = \\
\lim_{x \to \infty} \frac{6 \times 2 \log \varepsilon x \cdot \frac{1}{x}}{\frac{1}{2} \cdot x^{-\frac{1}{2}}} = \\
\lim_{x \to \infty} \frac{24 \log \varepsilon x}{x^{\frac{1}{2}}} = \text{de l' Hospital} \\
\lim_{x \to \infty} 24 \left( \frac{\log \varepsilon x}{x^{\frac{1}{2}}} \right) = \\
\lim_{x \to \infty} \frac{24 \cdot \frac{1}{x}}{\frac{1}{2} \cdot x^{-\frac{1}{2}}} = \\
\lim_{x \to \infty} 40 \frac{1}{x^{\frac{1}{2}}} = 0. \quad \text{Therefore}, \quad \log_3 x = 0 (\sqrt{x})
Example

Prove that \( x^2 = O(e^x) \)

Proof: We will use the theorem saying

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = c \Rightarrow f(x) = O(g(x))
\]

Let \( f(x) = x^2 \) and \( g(x) = e^x \)

\[
\lim_{x \to \infty} \frac{x^2}{e^x} = \text{de l'Hospital}
\]

\[
\lim_{x \to \infty} \frac{2x}{e^x} = \text{de l'Hospital}
\]

\[
\lim_{x \to \infty} \frac{2}{e^x} = 0
\]

Therefore, \( x^2 = O(e^x) \)
Example: Prove that $100x^3 + \log x + x^{\frac{1}{2}} = O(x^3)$

Proof: We will use the theorem saying that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = C \quad \Rightarrow \quad f(x) = O(g(x))$$

Let $f(x) = 100x^3 + \log x + x^{\frac{1}{2}}$

and $g(x) = x^3$

$$\lim_{x \to \infty} \frac{100x^3 + \log x + x^{\frac{1}{2}}}{x^3} = \text{de l'Hospital}$$

$$\lim_{x \to \infty} \left( 100x^3 + \log x + x^{\frac{1}{2}} \right)' = \frac{(x^3)'}{(x^3)}'$$

$$\lim_{x \to \infty} \frac{100 \times 3x^2 + \frac{1}{x} + \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}}}{3x^2}$$
\[ \lim_{x \to \infty} \left( \frac{300}{3} \frac{x^2}{x^2} + \frac{1}{x} \frac{1}{3} + \frac{1}{2} \frac{1}{3x^2} \right) = \]

\[ \lim_{x \to \infty} \left( 100 + \frac{1}{3x^3} + \frac{1}{6x^{2.5}} \right) = \]

100, which is a constant.

Therefore, \(100x^3 + \log x + x^{\frac{1}{2}} = O(x^3)\)