LP Rounding
A factor 2 Approximation for WEIGHTED VERTEX COVER

G(V,E) simple undirected graph.
\[ \text{w: } V \rightarrow R^+ \text{ a weight/cost function on its vertices} \]
Want a set of vertices of minimum total cost such that every edge is incident to a picked vertex.

Clearly a generalization of VERTEX COVER for which we know a factor 2 approximation algorithm via maximal matching.

Clearly the maximal matching approach will not suffice to approximate WEIGHTED VERTEX COVER.
Let us write the problem as an integer program:

(IP)

\[
\begin{align*}
& \min \sum_{v \in V} x_v w_v \\
& \text{s.t. } x_u + x_v \geq 1, \quad uv \in E \\
& \quad x_v \in \{0,1\}, \quad v \in V
\end{align*}
\]

Let us write the LP-relaxation of this integer program:

(LP)

\[
\begin{align*}
& \min \sum_{v \in V} x_v w_v \\
& \text{s.t. } x_u + x_v \geq 1, \quad uv \in E \\
& \quad x_v \geq 0, \quad v \in V
\end{align*}
\]

Note: Do not need to write \( x_v \leq 1 \) since satisfying (A) will never yield an \( x_v > 1 \), and it is a minimization problem.

Let \( x^* \), \( v \in V \), be an optimal solution to (LP). We know that we can find this in polynomial time, it is no more expensive than an optimal solution to (IP), but it is **FRACTIONAL**/NOT **INTEGRAL**

Claim: For each edge \( uv \), either \( x_v^* \geq \frac{1}{2} \) or \( x_u^* \geq \frac{1}{2} \) (or both)

Proof: Follows from the fact that \( x^* \) satisfies constraint (A).

Claim/Algorithm

\[
x_v = \begin{cases} 
1 & \text{if } x_v^* \geq \frac{1}{2} \\
0 & \text{if } x_v^* < \frac{1}{2}
\end{cases}
\]

This is a **Rounding Algorithm** and it is clearly an integral solution to (IP).
THEOREM  (factor 2 performance guarantee)

\[ \sum_{v \in V} x_v w_v \leq 2 \text{OPT (IP)} \]

PROOF

\[ \sum_{v \in V} x_v w_v = \sum_{v \in V} w_v \]  
(by definition of \(x_v\)'s in algorithm)

\[ \leq \sum_{v \in V} 2x_v^* w_v \]  
(1 \leq 2 \text{ for } x_v^* \geq \frac{1}{2})

\[ \leq 2 \sum_{v \in V} x_v^* w_v \]  
(\text{just adding more terms})

\[ = 2 \text{OPT (LP)} \]

\[ \leq 2 \text{OPT (IP)} \]