A LINEAR TIME ALGORITHM TO DETECT A MAJORITY ELEMENT

Let \( A = a_0 \ldots a_{n-1} \) be a list of \( n \) elements.

Case 1: \( n \) is even
Let us pair the elements \((a_0, a_1), \ldots, (a_{2i}, a_{2i+1}), \ldots, (a_{n-2}, a_{n-1})\)
Define the list \( B = b_1 \ldots b_{\frac{n}{2}} \) as follows:
\[
b_i = \begin{cases} a_{2i} & \text{if } a_{2i} = a_{2i+1} \\ \text{nil} & \text{if } a_{2i} \neq a_{2i+1} \end{cases}
\]
Let \( K \leq \frac{n}{2} \) be the number of non-nil elements of \( B \).

Claim 1: If a non-nil element \( x \) appears \( K_0 \leq K \) times in \( B \), then \( x \) appears at most \( 2K_0 + (\frac{n}{2} - K) \) times in \( A \).

Proof: There are exactly two appearances of \( x \) in \( A \) for each appearance of \( x \) in \( B \), plus at most one appearance of \( x \) in \( A \) for each nil element of \( B \) (meaning that one of the two elements that resulted in the nil pair was actually \( x \)).

Claim 2: If \( x \) is a majority element in \( A \), then \( x \) appears at least once in \( B \).

Proof: By the pigeonhole principle, if \( x \) appears more than \( \frac{n}{2} \) times in \( A \), then at least one of the pairs \((a_{2i}, a_{2i+1})\) has \( a_{2i} = a_{2i+1} = x \).
Claim 3. Let \( x \) be a majority element in \( A \) and suppose that \( x \) appears in \( B \) \( K_0 \geq 1 \) times (by Claim 2). Then \( x \) is a majority element among the \( K \) non-nil elements of \( B \), i.e. \( K_0 > K - K_0 \).

Proof: By contradiction. Assume \( K_0 \leq K - K_0 \).

Now combine this with Claim 1, indicating that \( K_0 \) appearances of \( x \) in \( B \) imply at most
\[
2K_0 + \left( \frac{n}{2} - K \right)
\]
appearances of \( x \) in \( A \).

And since \( K_0 \leq K - K_0 \Rightarrow -K \leq -2K_0 \),

This is a total of at most \( 2K_0 + \frac{n}{2} - 2K_0 = \frac{n}{2} \)
appearances of \( x \) in \( A \), implying that \( x \) is not a majority element of \( A \) which is a contradiction.

Note 1. A majority among the non-nil elements of \( B \) is not necessarily a majority among the elements of \( A \).

Example \( A = \text{aaa aa bc de} \) but \( a \) is not a majority element of \( A \).

The above imply that majority elements "propagate", but not everything that "propagates" is necessarily majority in the first place.
Case 2 \( n \) is odd \( j \) we cannot do the "pairing" an\(-1\)

In this case we can just single out one element of \( A \) and do the following:
- Test if \( y \) is majority by examining \( a_0 \ldots a_{n-1} \)
- If the answer is YES then we are done
- If the answer is NO then we can ignore \( a_{n-1} = y \)
  and be left with the even length list
  \[ A' = a_0 \ldots a_{n-2} \]

Claim 4 \( x \) is a majority element of \( A = a_0 \ldots a_{n-1} \), \( n \) is odd,
  and \( a_{n-1} = y \neq x \)
  then \( x \) is a majority element of \( A' = a_0 \ldots a_{n-2} \)
Majority $^*$ $(a_0, \ldots, a_{n-1})$

if $n=1$ then return $(a_0)$
if $n>1$ then

if $n$ is odd then

$$k = \left\lfloor \sum_{i=0}^{n-1} a_i = \frac{n}{2}\right\rfloor$$
if $k > n-k$ then return $(a_{n-1})$
if $k \leq n-k$ then $\text{Majority}^* (a_0, \ldots, a_{n-2})$

if $n$ is even then

$$k = 0$$
for $i = 0$ to $\frac{n}{2}-1$

if $a_{2i} = a_{2i+1}$ then

$$b_k = a_{2i}$$
k := k + 1

if $k > 0$ then $\text{Majority}^* (b_0, \ldots, b_k)$
if $k = 0$ then return (nil)

THEOREM

$\text{Majority}^*$ runs in linear time:

$T(n) = T\left(\lceil \frac{n}{2} \rceil \right) + O(n)$

If $a_0, \ldots, a_{n-1}$ have a majority element $x$ then $\text{Majority}^*$ will return $x$

THEOREM

$\text{Majority}$ runs in linear time and returns $x \neq \text{nil}$ if and only if $x$ is a majority element of $a_0, \ldots, a_{n-1}$

Majority $(a_0, \ldots, a_{n-1})$

$x := \text{Majority}^* (a_0, \ldots, a_{n-1})$

if $x \neq \text{nil}$ then

$$\left\lfloor \sum_{i=0}^{n-1} a_i = x \frac{n}{2}\right\rfloor > \left\lfloor \sum_{i=0}^{n-1} a_i = x \frac{n}{2}\right\rfloor$$
then

return $(x)$

else return $(\text{nil})$

if $x = \text{nil}$ then return $(\text{nil})$