"Virtual" lecture on Randomization

See CMU lecture notes on Randomized Median Finding,

including the analysis of the Expected Running Time
The randomized median finding algorithm is a Las Vegas algorithm. It always terminates with the correct answer, and its expected running time is $T(n) = cn \log n$ for some constant $c$.

Characterizing only the expectation of a random variable is not fully satisfactory. Need further: How the random variable is concentrated around its mean.

From an algorithmic efficiency point of view we want to characterize the probability that a randomized algorithm fails to terminate within a certain amount of time.
Monte Carlo Median

\text{MEDIAN} := \text{UNKNOWN}

\text{for } i := 1 \text{ to } K

\text{run Las Vegas Rand Median for } 2T(n) = 2cn \log n \text{ steps}

\text{if median was found then } \text{MEDIAN} := \text{found median}

\text{if } \text{MEDIAN} = \text{UNKNOWN} \text{ then return (FAILED)}

\text{else return (MEDIAN)}

Main Theorem

The running time of Monte Carlo Median is \(2KT(n)\).

The probability that Monte Carlo Median outputs \text{FAILED}

is at most \(\frac{1}{2K}\)

Proof of Main Theorem on Page 5
Markov's Inequality

Let $X$ be a non-negative random variable.

Let $E(X) > 0$ be its expectation.

Let $\lambda > 1$.

$$\Pr \left[ X > \lambda E[X] \right] \leq \frac{1}{\lambda}$$

**Proof** (formalizes absolute basic common sense)

Assume for contradiction that $\Pr \left[ X > \lambda E[X] \right] > \frac{1}{\lambda}$

Then:

$$E[X] = \sum_x x \Pr \left[ X = x \right]$$

$$= \sum_{x \leq \lambda E[X]} x \Pr \left[ X = x \right] + \sum_{x > \lambda E[X]} x \Pr \left[ X = x \right]$$

$$\leq 0 + \lambda E[X] \sum_{x > \lambda E[X]} \Pr \left[ X = x \right]$$

$$= \lambda E[X] \Pr \left[ X > \lambda E[X] \right]$$

$$< \lambda E[X] \frac{1}{\lambda} = E[X]$$

Thus $E[X] < E[X]$, contradiction.
Proof of Main Theorem:

From Markov's inequality, the probability that the Las Vegas Randomized Median algorithm runs for time $> 2T(n) = 2cn \log n$

is $\leq \frac{1}{2}$

Monte Carlo Median outputs FAILED

iff Las Vegas Randomized Median did not terminate in $2T(n)$ steps $K$ times in a row.

This probability is $\leq \frac{1}{2^K}$