1 Traveling Salesman Problem

- $G = (V, E)$ is a complete undirected graph
- Non-negative integer cost $c(u, v)$ with each edge $(u, v) \in E$.
- **TSP**: Find a Hamiltonian cycle of $G$ with minimum cost.
- **TSP with triangle inequality**: The cost function $c$ satisfies the triangle inequality if for all vertices $u, v, w \in V$:
  \[ c(u, w) \leq c(u, v) + c(v, w). \]
- TSP with triangle inequality is NP-Complete: also known as metric TSP or constrained TSP.
  [Proved in HW]
- The TSP has several applications in planning, logistics, VLSI design, DNA sequencing etc.
- Probably the most well-studied problem in combinatorial optimization.

2 Inapproximability of TSP

**Claim**: If $P \neq \text{NP}$, then for any polynomial time computable function $\rho(n)$, there is no polynomial time $\rho(n)$-approximation algorithm for the general TSP.

- Suppose that there is a polynomial time $\rho$ approximation algorithm, say $A$ for TSP.
- We will show that we can use $A$ to decide the Hamiltonian Cycle problem which is NP-complete thus showing that $P = \text{NP}$.
  - Let $G = (V, E)$ be an undirected graph.
  - Construct a complete graph $G' = (V, E')$ from $V$.
  - For each $u, v \in E'$, assign an integer cost:
    * $c(u, v) = 1$ if $(u, v) \in E$ and
    * $c(u, v) = \rho \times |V| + 1$ if $(u, v) \notin E$.
  - Run $A$ on $G'$ with this cost function on the edges.
  - Suppose $G$ has a Hamiltonian cycle.
    * The cost of this cycle in $G'$ is $|V|$.
    * $A$ returns a tour whose cost is at most $\rho \times |V|$.
  - Suppose $G$ has no Hamiltonian cycle.
    * The cost of any Hamiltonian cycle in $G'$ is $> \rho \times |V|$:
      - Any Hamiltonian cycle in $G'$ must include an edge not in $E$.
      - Any Hamiltonian cycle has cost at least $(\rho \times |V| + 1) + (|V| - 1)$ which is $> \rho \times |V|$.
3 A 2-approximation algorithm for metric TSP

1. Construct a minimum spanning tree (MST) $T$.
2. Double every edge of $T$ to get an Eulerian graph.
3. Find an Eulerian tour $W$ on this graph. We can take a preorder traversal of $T$.
4. Let $L$ be the list of vertices obtained by deleting all duplicates in $W$ by keeping, for all vertices $u$, only the first visit to the vertex $u$.
5. Let $H$ be the cycle corresponding to this traversal.

4 Analysis

Claim: The algorithm given above is a 2-optimal approximation algorithm.

- Let $H^*$ be an optimal TSP tour.
- Then, $C(T) \leq C(H^*)$.
  - Deleting an edge from $H^*$ gives a spanning tree of $G$.
- Let $W$ be a list of vertices from a preorder traversal of $T$ before removing duplicates.
- Then, $C(W) = 2C(T)$:
  - Every edge of $T$ is traversed exactly twice in $W$.
- Therefore, $C(W) \leq 2C(H^*)$.
- Let $H$ be the cycle obtained by deleting all duplicates in $W$ by keeping, for all vertices $u$, only the first visit to the vertex $u$.
- Then, $C(L) \leq C(W)$:
  - Let $W'$ be the list obtained from $W$ after the deletion of some vertices.
  - Say a vertex $v$ occurring in the order $u,v,w$ in $W'$ is deleted.
  - Then, the cost of the resulting list is at most the cost of $W'$:
    - There is an edge between $u$ and $w$ since $G$ is complete.
    - By triangle inequality, $c(u,w) \leq c(u,v) + c(v,w)$.
- Exercise: The analysis is tight!
5 Christofides Algorithm: 3/2 approximation for metric TSP

1. Construct a minimum spanning tree $T$.
2. Compute a minimum cost perfect matching $M$ on the set of odd-degree vertices of $T$. Add $M$ to obtain an Eulerian graph.
3. Find an Eulerian tour $W$ on this graph.
4. Let $L$ be the list of vertices obtained by deleting all duplicates in $W$ by keeping, for all vertices $u$, only the first visit to the vertex $u$.
5. Let $H$ be the cycle corresponding to this traversal.

6 Analysis

• Key idea: Use perfect matching in odd degree vertices of MST to obtain an Eulerian graph in step 2.
• Let $S \subseteq V$ and $|S|$ is even and $M$ is a minimum cost perfect matching on $S$ then $\text{cost}(M) \leq \text{Opt}/2$
  
  – Let $H^*$ be the optimal TSP tour and $\text{cost}(H^*) = \text{Opt}$
  – Let $H'$ be the tour on $S$ by short-cutting $H^*$.
  – By triangle inequality, $\text{cost}(H') \leq \text{Opt}$.
  – Now $H'$ is union of two perfect matchings on $S$.
  – The cheaper of these two matchings has cost $\leq \text{cost}(H')/2 \leq \text{Opt}/2$.
• $\text{cost}(H) \leq \text{cost}(T) + \text{cost}(M) \leq \text{Opt} + \text{Opt}/2 \leq 3/2\text{Opt}$.
• The Analysis is tight!
• Exercise: Find such a tight example.

7 Other Comments:

• It is a BIG open question in theoretical computer science to get a $3/2 - \epsilon$ approximation for metric TSP for any $\epsilon > 0$.
• The Euclidean TSP, or planar TSP, is the TSP with the distance being the ordinary Euclidean distance.
• The Euclidean TSP is a particular case of the metric TSP, since distances in a plane obey the triangle inequality.
• Sanjeev Arora and Joseph S. B. Mitchell were awarded the Gödel Prize in 2010 for their concurrent discovery of a PTAS for the Euclidean TSP.
• There are commercial softwares like Concorde which can solve most of the problems with millions of cities within a small fraction of 1% of the optimal.
8 Resources:

I am following chapter 2.4 (The traveling salesman problem) of [1] for the lectures. The book is freely available online: [http://www.designofapproxalgs.com/](http://www.designofapproxalgs.com/). You can also see chapter 3 (Steiner Tree and TSP) from [2].

References
