Efficiency of (Revenue-)Optimal Mechanisms

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ABSTRACT

We compare the expected efficiency of revenue maximizing (or *optimal*) mechanisms with that of efficiency maximizing ones. We show that the efficiency of the revenue maximizing mechanism for selling a single item with $(k + \log_{\frac{e}{e-1}} k + 1)$ bidders is at least as much as the efficiency of the efficiency maximizing mechanism with k bidders, when bidder valuations are drawn i.i.d. from a Monotone Hazard Rate distribution. Surprisingly, we also show that this bound is tight within a small additive constant of 4.7. In other words, $\Theta(\log k)$ extra bidders suffice for the revenue maximizing mechanism to match the efficiency of the efficiency maximizing mechanism, while $o(\log k)$ do not. This is in contrast to the result of Bulow and Klemperer [1] comparing the revenue of the two mechanisms, where only one extra bidder suffices. More precisely, they show that the revenue of the efficiency maximizing mechanism with $k + 1$ bidders is no less than the revenue of the revenue maximizing mechanism with k bidders.

We extend our result for the case of selling t identical items and show that $\Theta(\log k) + t\Theta(\log \log k)$ extra bidders suffice for the revenue maximizing mechanism to match the efficiency of the efficiency maximizing mechanism.

In order to prove our results, we do a classification of Monotone Hazard Rate (MHR) distributions and identify a family of MHR distributions, such that for each class in our classification, there is a member of this family that is pointwise lower than every distribution in that class. This lets us prove interesting structural theorems about distributions with Monotone Hazard Rate.

Categories and Subject Descriptors

F.m [**Theory of Computation**]: Miscellaneous

General Terms

Design, Economics, Theory

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Auction Design, Efficiency, VCG, Optimal Auctions

1. INTRODUCTION

Auctions are bid-based mechanisms for buying and selling of goods. The two most common objective functions in auction design are efficiency and revenue. Efficiency is the sum of the surplus of both the seller and the buyer, which represents the total social welfare, whereas revenue is the surplus of the seller only. The efficiency and revenue of auctions has been the subject of extensive study in auction theory (see, e.g., the survey [9], and the citations within). Ideally we would like to simultaneously maximize both the objective functions. But these two goals cannot be achieved simultaneously. Thus, we have the celebrated Vickrey-Clarke-Groves (VCG) mechanism [8, 2, 3] which is a truthful mechanism that maximizes efficiency on the one hand, and Myerson's Optimal Mechanism [5] that maximizes revenue on the other. If one is interested in both objective functions, then one has to trade-off the two. The exact balance of how much weight to give to each objective is, perhaps, a difficult question in any real-world setting. Generally, private sellers would like to maximize revenue, but keeping long-term benefit of the business in mind, they might want to keep social welfare high as well. Similarly, the primary goal in allocating public goods is to maximize social welfare, but a secondary objective might be to raise revenue.

In the light of this dilemma, a natural question that arises is: how sub-optimal is the revenue of the VCG mechanism, and how sub-optimal is the efficiency of Myerson's Optimal Mechanism. Bulow and Klemperer [1] give a structural theorem which characterizes the sub-optimality of the revenue of the VCG mechanism. They show that for the case of selling a single item, the VCG mechanism with one extra bidder makes at least as much revenue in expectation as the expected revenue of Myerson's Optimal mechanism (when bidder valuations are drawn from a class of distributions called regular distributions). A seller, who is currently using the VCG mechanism and wishes to increase her revenue, faces two choices: (1) increase the reserve price closer to Myerson's reserve price, or (2) attract more bidders by investing in sales and marketing. Bulow and Klemperer's theorem gives an insight into the trade-offs between these two choices.

In this paper, we characterize the sub-optimality of the efficiency of Myerson's Optimal mechanism. We show that, surprisingly, there exists a class of distributions with monotone hazard rate for which a constant number of extra bidders does not suffice for Myerson's optimal mechanism to match the efficiency of the VCG mechanism. In fact, we show that one needs at least $\omega(\log k)$ extra bidders, where k is the number of bidders participating in the VCG mechanism. We match this lower bound by showing that, for all monotone hazard rate distributions, $O(\log k)$ extra bidders always suffice (our upper and lower bounds are tight up to a small additive constant). This contradicts the following intuition: since the efficiency of Myerson's Optimal mechanism gets closer to the VCG mechanism as the number of bidders k increases, we would expect the number of extra bidders needed would go down as k increases. Another way of interpreting our upper bound is that with $O(\log k)$ extra bidders, Myerson's Optimal mechanism simultaneously maximizes both revenue and efficiency. We extend our upper bound result to the case of selling multiple identical items as well.

In order to prove the above results, we do a classification of Monotone Hazard Rate (MHR) distributions and identify a family of MHR distributions, such that for each class in our classification, there is a member of this family that is pointwise lower than every distribution in that class. This enables us to prove certain important structural properties of distributions with Monotone Hazard Rate, which helps us prove our main theorems.

1.1 Model

Our model consists of a single seller and k buyers (bidders). We will consider the case of selling a single item, and also the case of selling t identical items when each bidder has unit demand. The private values $(v_i)_{i\in[k]}$ of the bidders are drawn independently from a common distribution D. Here v_i represents the value of the bidder i for one unit of the item. We will use f_D and F_D to denote the probability density function and cumulative distribution function of the distribution D respectively.

The *hazard rate* of a distribution D is given by $h_D(x) := f_D(x)/(1 - F_D(x))$. A distribution D is said to have a *monotone hazard rate* (MHR) if $h_D(.)$ is a non-decreasing function of x. For the most part, we will assume (as is common in the economics literature) that the given distribution D has a monotone hazard rate. Many common families of distributions such as the Uniform and the Exponential families have MHR. We will assume that the support of D lies between $[0, \infty)$. Also, for the ease of presenting main ideas, we will assume that F_D is a continuous function even though the results hold in non-continuous case as well.

We will restrict our attention to truthful auctions, those in which bidders have no incentive to misreport their true valuation. Thus we can assume that the bidders bid their true private values, i.e., the bid vector $(b_i)_{i\in[k]}$ is same as the value vector $(v_i)_{i\in[k]}$. From here onwards, we will use the terms *bid* and *value* interchangeably.

For a given mechanism, its *efficiency* on a given input is defined as the sum of the valuations of the bidders who get the good, while its *revenue* is defined as sum of the payments to the seller. Since the private values of bidders are not arbitrary but rather drawn from a distribution, we will be interested in the values of efficiency and revenue *in expectation.* We will use $\textsc{Eff}(M)$ to denote the expected efficiency of a mechanism M.

1.2 Optimal Auctions: Vickrey and Myerson

We will use $\textsc{Ema}(k)$ to denote the efficiency maximizing (VCG) auction with k bidders (D will always be clear from context). It assigns the item to the highest bidder and charges it the second highest bid. In the case of selling t identical items, $\textsc{Ema}(k)$ allocates the items to the t highest bidders, charging each of them the $t + 1$th highest bid.

Similarly, we will use $\textsc{Rma}(k)$ to denote revenue maximizing auction (Myerson's auction) with k bidders. Myerson [5] defined a notion of *virtual valuation* ψ_i of a bidder i, where

$$\psi_i := v_i - \frac{1}{h_D(v_i)}$$

Myerson showed that the revenue maximizing truthful auction is the one which maximizes the virtual efficiency (sum of the virtual valuations of the auction winners). Thus, in the single item case, it assigns the item to the bidder with the highest non-negative virtual value (and does not sell the item if all virtual values are negative). If a distribution D satisfies the *regularity* condition, defined as $\psi(x)$ being a non-decreasing function, then the above condition is equivalent to assigning item to the highest bidder as long his virtual value is non-negative (note that distributions with MHR always satisfy the regularity condition). This cutoff value at which $\psi(x) = 0$ is called the reserve price r_D.

Definition 1 (Reserve price). *Reserve price of distribution D is defined as:*

$$r_D := x, \quad s.t. \quad h_D(x) = 1/x$$

We drop the subscript D, if D is clear from context. Thus in the single item case with regular distributions, $\textsc{Rma}(k)$ assigns the item to the highest bidder, as long as its bid is no smaller than the reserve price, and charges it the maximum of the reserve price and the second highest bid (it does not sell the item if all bids are below the reserve price). In the case of selling t identical items, $\textsc{Rma}(k)$ finds the $k' \le k$ bidders whose bids are above the reserve price, allocates one item each to the highest $\min\{t, k'\}$ bidders and charges each one the maximum of the reserve price and the $t+1$th highest bid.

1.3 Related Work

Bulow and Klemperer [1] characterized the revenue suboptimality of $\textsc{Ema}$. They showed that $\textsc{Ema}(k + 1)$ (with one extra bidder) has at least as much expected revenue as $\textsc{Rma}(k)$. Their result can be interpreted in a bi-criteria sense; VCG auctions with one extra bidder simultaneously maximize both revenue and efficiency. For the case of t identical items, they show that t additional bidders are needed for the result to hold.

In [7], Roughgarden and Sundararajan gave the *approximation factor* of the optimal revenue that is obtained by $\textsc{Ema}(k)$. They show that, for t identical items and k bidders with unit demand, the revenue of $\textsc{Ema}(k)$ is at least $(1 - t/k)$ times the revenue of $\textsc{Rma}(k)$. Neeman [6] also studied the percentage of revenue which $\textsc{Ema}(k)$ makes compared to $\textsc{Rma}(k)$ in the single item case. [6] used a numerical analysis approach and assumed that the distribution D is any general distribution (not restricted to regular or MHR as in [7]) but with a bounded support.

In another related work looking at simultaneously optimization of both revenue and efficiency, Likhodedov and

Sandholm [4] gave a mechanism which maximizes efficiency, given a lower bound constraint on the total revenue.

1.4 Our Results

We study the number of extra bidders required for Myerson's (revenue-)optimal mechanism to achieve at least as much efficiency as the efficiency maximizing (VCG) mechanism. Let $\alpha = 1 - 1/e$. For bidder valuations drawn from a distribution with Monotone Hazard Rate, we prove the following:

- *Single item Upper Bound (Theorem 8):* In the single item case, $m \geq \left\lfloor \log_{\frac{1}{\alpha}} 2k \right\rfloor + 2$ extra bidders suffice to for the revenue maximizing mechanism to achieve at least as much efficiency as the efficiency maximizing mechanism with k bidders (for any k).
- *Single item Lower Bound (Theorem 9):* In the single item case, we demonstrate a distribution having monotone hazard rate, such that for any k, if the efficiency-optimal mechanism has k bidders, and the revenue-optimal mechanism has $\left\lfloor \log_{1/\alpha} (k+1)(1-\alpha) \right\rfloor + 1$ extra bidders, then the efficiency of the latter is strictly less than the efficiency of the former. In other words, $\left\lfloor \log_{1/\alpha} (k+1)(1-\alpha) \right\rfloor + 1$ extra bidders do not suffice.
- *Multi item Upper Bound (Theorem 11):* In the case of selling t identical items, with bidders having unit-demand: $m + s$ extra bidders suffice for the revenue-optimal mechanism to achieve at least as much efficiency as the efficiency-optimal mechanism with k bidders, where $m = \left\lfloor \log_{\frac{1}{\alpha}} 2k \right\rfloor + 2$ and $s \geq t + (1 + \epsilon)t \log m$, for every $\epsilon > 0$ and large enough k. Thus, approximately, $\log k + t \log \log k$ extra bidders suffice.
- We also show that if both auctions have the same number of bidders k, then the ratio of the efficiency of the revenue-optimal auction to the optimal efficiency is at least $1 - \alpha^k$. (The proof is easy and we omit it in this extended abstract[1]).
- We also prove (Section 5) that our upper bound result does not hold for regular distributions – we show that for every k, m, there is a regular distribution for which the efficiency of the revenue-optimal mechanism with $k + m$ bidders is strictly lower than the efficiency of the efficiency-optimal mechanism with k bidders.

Outline of the paper:

Section 2 describes some basic setup which is common to the rest of the paper. Section 3 describes our results for the single item case. We start with the definition of two quantities Gain and Loss, and analyze the expression Gain − Loss in Section 3.1. Section 3.2 and 3.3 prove our upper and lower bound results respectively. In Section 4, we extend out upper bound result to the case of selling t indentical items to bidders with unit-demand. Section 5 deals with the case of regular distributions.

[1] In fact we can also prove using the same techniques that the ratio of the revenue of VCG to that of Myerson's Optimal Auction is $1 - \alpha^{k-1}$, which improves on the polynomial bound provided in [7].

2. BASIC SETUP

To compare the two mechanisms Rma($k+m$) and Ema(k), which have a different number of bidders, we will think of the process of drawing their bidder values as drawing $k + m$ bidder values $(b_1, b_2, ..b_{k+m})$ independently from the distribution D – the first k bidders $(b_1, b_2, ..b_k)$ participate in both Rma($k + m$) and Ema(k), whereas the last m bidders participate only in Rma($k + m$).

The following lemma will prove useful in the subsequent sections.

Lemma 1. *For any MHR distribution D: $F_D(r) \leq 1 - 1/e$, where r is the reserve price for the distribution D.*

Proof. Let the hazard rate of distribution D be $h(x)$. By definition of hazard rate, we have $F_D(x) = 1 - e^{-\int_0^x h(t)dt}$. By definition of reserve price, $h(r) = 1/r$. Also since D has MHR, we get $h(x) \leq 1/r$, $\forall x \leq r$. Thus,

$$\begin{aligned} \forall x \leq r: \quad 1 - e^{-\int_0^x h(t)dt} &\leq 1 - e^{-\int_0^x 1/rdt} \\ \Rightarrow F_D(r) &\leq 1 - e^{-\int_0^r 1/rdt} \\ &\qquad \text{(setting } x = r\text{)} \\ \Rightarrow F_D(r) &\leq 1 - e^{-1} \end{aligned}$$

□

3. SELLING ONE ITEM

We begin by noting that when selling a single item, if the value of any of the first k bidders is greater than or equal to the reserve price r_D, then Rma($k + m$) achieves at least as much efficiency as Ema(k).

The more challenging case (for the upper bound) occurs when the value drawn by all the first k bidders is less than the reserve price. In this case, Ema(k) achieves an efficiency equal to the highest value among the values of the first k bidders, whereas the contribution of the first k bidders to Rma($k + m$) is zero as all of these bidders have a value less than the reserve price cutoff. In other words, conditioned on the event that first k bidders' value is less than the reserve price r_D, the expected efficiency of Ema(k) is $\frac{\int_0^r x f^{(k)}(x)dx}{F^{(k)}(r)}$, where $F_D^{(k)}$ and $f_D^{(k)}$ are the c.d.f and p.d.f of the maximum of k numbers picked i.i.d. from D (we will drop the subscript D whenever D is clear from the context). Note that $F^{(k)}(x) = F^k(x)$, and $f^{(k)}(x) = kF^{k-1}(x)f(x)$. Also, conditioned on this event, the expected contribution of the first k bidders $(b_1, b_2, ..., b_k)$ to the efficiency of Rma($k + l$) is zero. We define:

$$\text{Loss}_D = \frac{\int_0^r x f_D^{(k)}(x)dx}{F_D^k(r)} \tag{1}$$

To make up for this lost efficiency, the revenue maximizing mechanism has m extra bidders. The contribution to the expected efficiency of Rma($k + m$) from these m extra bidders $(b_{k+1}, b_{k+2}, ..b_{k+m})$, conditioned on the event that the value of bidders $(b_1, b_2, ..b_k)$ is less than r_D, is at least $(1 - F^m(r_D))r_D$. This is because of the fact that all the draws are independent and the probability that at least one of the m extra bidders will have a value higher than the reserve price r_D is $(1 - F^m(r_D))$. We define:

$$\text{Gain}_D = (1 - F^m(r_D))r_D \tag{2}$$

If for all distributions D, it is true that $\text{GAIN}_D - \text{LOSS}_D \geq 0$ for some k, m, then we know that the efficiency of $\text{RMA}(k+m)$ is at least as much as the efficiency of $\text{EMA}(k)$. Moreover, if we can demonstrate a distribution D s.t. the expected contribution to the efficiency of $\text{RMA}(k+m)$ from these extra m bidders $(b_{k+1}, b_{k+2}, ..b_{k+m})$, is strictly less than GAIN_D, i.e., if we show that $\text{GAIN}_D - \text{LOSS}_D < 0$, for some k, l, (and that there is no additional gain to $\text{RMA}(k+m)$ in the case when one of the first k bidders has value equal to or greater than r_D), then we would have shown that m extra bidders does not suffice.

Therefore, we examine this key expression $\text{GAIN}_D - \text{LOSS}_D$ in the next section. We will omit the subscript D whenever it is clear from the context.

3.1 The expression GAIN − LOSS

Recall the expressions GAIN_D and LOSS_D as defined in Equations 2 and 1.

For the purpose of getting a better handle on the expression GAIN − LOSS, we will partition the set of all MHR distributions into different classes, according to their optimal reserve price r and the value of the cdf at the optimal reserve price, as follows. Let $\mathcal{D}(r, \phi)$ be the set of MHR distributions with a fixed reserve price $r \geq 0$ and $F(r) = \phi$. Note that, by Lemma 1, $\mathcal{D}(r, \phi)$ is non-empty only if $\phi \in [0, 1 - 1/e]$. Also note that all these distributions have the same value for the expression GAIN and differ only in the numerator of the expression LOSS.

Next, we find a distribution in $\mathcal{D}(r, \phi)$ which maximizes the numerator of LOSS.

DEFINITION 2 (DISTRIBUTION $G_{\phi,r}$). *Let* $r \geq 0$, $\phi \in [0, 1-1/e]$, *and let* $t(\phi, r) = r(1 + \ln(1 - \phi)) \in [0, r]$. *Then,*

$$G_{\phi,r}(x) = \begin{cases} 0 & 0 \leq x < t(\phi, r), \\ 1 - e^{-\frac{1}{r}(x - t(\phi,r))} & t(\phi, r) \leq x \leq r, \\ \phi + \frac{1-\phi}{\epsilon}.(x - r) & r \leq x \leq r + \epsilon, \\ 1 & x \geq r + \epsilon. \end{cases}$$

Here ϵ is any positive number. It can be verified that the above distribution has the following properties:

- $G_{\phi,r}$ is MHR.
- The optimal reserve price for $G_{\phi,r}$ is r.
- $G_{\phi,r}(r) = \phi$. Therefore, $G_{\phi,r} \in \mathcal{D}(r, \phi)$.

Next, we prove that $G_{\phi,r}$ is (point-wise) no larger than every other function in $\mathcal{D}(r, \phi)$

LEMMA 2. *For every distribution* $D \in \mathcal{D}(r, \phi)$, *and any* $y \in [0, r]$*:*

$$F_D(y) \geq G_{\phi,r}(y)$$

PROOF. The cdf of a distribution D with hazard rate $h_D(.)$ can be written as $F_D(y) = 1 - e^{-\int_0^y h_D(z)dz}$.

Now, by definition, $r - \frac{1}{h_D(r)} = 0$, implying $h_D(r) = 1/r$. Since $h_D(.)$ is an increasing function, therefore $h_D(z) \leq 1/r$ for all $z \leq r$. Also note that $h_{G_{\phi,r}}(z) = 0$ for $z < t(\phi, r)$, and $h_{G_{\phi,r}}(z) = 1/r$ for $z \in [t(\phi, r), r]$.

Thus, for any $y \in [t(\phi, r), r]$ we have

$$\int_y^r h_D(z)dz \leq \int_y^r h_{G_{\phi,r}}(z)dz$$

$$\Rightarrow \int_0^r h_D(z)dz - \int_0^y h_D(z)dz \leq \int_0^r h_{G_{\phi,r}}(z)dz - \int_0^y h_{G_{\phi,r}}(z)dz$$

Since $F_D(r) = G_{\phi,r}(r) = \phi$, (and recalling from the definition of hazard rate that $F_D(x) = 1 - e^{-\int_0^x h_D(t)dt}$ for all distributions D) we have $\int_0^r h_D(z)dz = \int_0^r h_{G_{\phi,r}}(z)dz$. Therefore,

$$-\int_0^y h_D(z)dz \leq -\int_0^y h_{G_{\phi,r}}(z)dz$$

$$\Rightarrow \quad 1 - e^{-\int_0^y h_D(z)dz} \geq 1 - e^{-\int_0^y h_{G_{\phi,r}}(z)dz}$$

Thus, for any $y \in [t(\phi, r), r]$, $F_D(y) \geq G_{\phi,r}(y)$. Since $G_{\phi,r}(y) = 0$ for $y \in [0, t(\phi, r))$, we have proved the lemma. □

Next we prove that the numerator of LOSS among $\mathcal{D}(r, \phi)$ is maximized at $G_{\phi,r}$.

LEMMA 3. *For every distribution* $D \in \mathcal{D}(r, \phi)$*:*

$$\int_0^r x f_D^{(k)}(x)dx \leq \int_0^r x[G_{\phi,r}^{(k)}(x)]'dx$$

PROOF. For any $D \in \mathcal{D}(r, \phi)$, we have

$$\begin{aligned} \int_0^r x f_D^{(k)}(x)dx &= rF_D^{(k)}(r) - \int_0^r F_D^{(k)}(x)dx \\ &\quad \text{(integrating by parts)} \\ &\leq rG_{\phi,r}^{(k)}(r) - \int_0^r G_{\phi,r}^{(k)}(x)dx \\ &= \int_0^r x[G_{\phi,r}^{(k)}(x)]'dx \end{aligned}$$

The inequality follows from Lemma 2 and the fact that $rF_D^{(k)}(r) = rG_{\phi,r}^{(k)}(r) = r\phi^k$. □

Now that we know that, among all distributions in $D(r, \phi)$, $G_{\phi,r}$ maximizes the expression LOSS, we can calculate the maximum value of LOSS in terms of r and ϕ. Lemma 6 examines the numerator of LOSS at $G_{\phi,r}$.

FACT 4. *For a fixed* λ, t, *and* k*:* $\int (1 - e^{-\lambda(x-t)})^k dx = x - \frac{1}{\lambda}\sum_{i=1}^k \frac{(1-e^{-\lambda(x-t)})^i}{i}$

PROOF. Let $g_k = \int (1 - e^{-\lambda(x-t)})^k dx$. We have

$$\begin{aligned} g_k &= \int (1 - e^{-\lambda(x-t)})^{k-1}dx \\ &\quad - \int e^{-\lambda(x-t)}(1 - e^{-\lambda(x-t)})^{k-1}dx \\ \Rightarrow \quad g_k &= g_{k-1} - \frac{1}{\lambda.k}(1 - e^{-\lambda(x-t)})^k \\ \Rightarrow \quad g_k &= g_0 - \frac{1}{\lambda}\sum_{i=1}^k \frac{(1 - e^{-\lambda(x-t)})^i}{i} \\ \Rightarrow \quad g_k &= x - \frac{1}{\lambda}\sum_{i=1}^k \frac{(1 - e^{-\lambda(x-t)})^i}{i} \end{aligned}$$

□

Corollary 5. *For a fixed r, t and k,* $\int_t^r (1-e^{-\frac{x-t}{r}})^k dx = r - t - r\sum_{i=1}^k \frac{\phi^i}{i}$, *where* $\phi = 1 - e^{-\frac{r-t}{r}}$.

Lemma 6. $\int_0^r x[G^{(k)}_{\phi,r}(x)]' dx$ $= r\left(\phi^k + \ln(1-\phi) + \sum_{i=1}^k \frac{\phi^k}{i}\right)$

Proof. Integrating by parts, we have

$$\begin{aligned}
\int_0^r x[G^{(k)}_{\phi,r}(x)]' dx &= r\phi^k - \int_0^r G^{(k)}_{\phi,r}(x)dx \\
&= r\phi^k - \int_{t(\phi,r)}^r (1 - e^{-\frac{1}{r}(x-t(\phi,r))})^k dx \\
&= r\phi^k - \left[r - t(\phi,r) - r\sum_{i=1}^k \frac{\phi^i}{i}\right] \\
&\qquad \text{(by Corollary 5)} \\
&= r\phi^k + r(1+\ln(1-\phi)) - r + r\sum_{i=1}^k \frac{\phi^i}{i} \\
&\qquad \text{(by definition of } t(\phi,r)) \\
&= r\left(\phi^k + \ln(1-\phi) + \sum_{i=1}^k \frac{\phi^i}{i}\right)
\end{aligned}$$

□

Let $D \in \mathcal{D}(r,\phi)$. Then, for any given m, from equations 2 and 1 and Lemma 6, we have:

$$\begin{aligned}
&\text{Gain}_D - \text{Loss}_D \\
&\geq \text{Gain}_{G_{\phi,r}} - \text{Loss}_{G_{\phi,r}} \\
&= (1-\phi^m)r - \frac{r\left(\phi^k + \ln(1-\phi) + \sum_{i=1}^k \frac{\phi^i}{i}\right)}{\phi^k} \\
&= \frac{r\left(-(\phi)^{k+m} - \ln(1-\phi) - \sum_{i=1}^k \frac{\phi^i}{i}\right)}{\phi^k} \qquad (3)
\end{aligned}$$

3.2 Upper bound on the number of extra bidders required

To prove an upper bound for distributions in $\mathcal{D}(r,\phi)$, we need to find values of m (as a function of k) for which the expression in Equation 3 is non-negative. To do this, we define a uni-variate polynomial

$$q(x) := x^{k+m} + \ln(1-x) + \sum_{i=1}^k \frac{x^i}{i}$$

Since $\phi \in [0, 1-1/e]$ by Lemma 1, it suffices to find m for which $q(x) \leq 0$ for all $x \in [0, 1-1/e]$.

Since $q(0) = 0$, if we find values of m for which $q'(x) \leq 0$, $\forall x \in [0, 1-1/e]$, then $q(x) \leq 0$ $\forall x \in [0, 1-1/e]$. Now,

$$\begin{aligned}
q'(x) &= (k+m)x^{k+m-1} - \frac{1}{1-x} + \sum_{i=1}^k x^{i-1} \\
&= (k+m)x^{k+m-1} - \frac{1}{1-x} + \frac{1-x^k}{1-x} \\
&= (k+m)x^{k+m-1} - \frac{x^k}{1-x} \\
&= \frac{x^k}{1-x}((k+m)x^{m-1}(1-x) - 1)
\end{aligned}$$

Since $\frac{x^k}{1-x} \geq 0$, $q'(x) \leq 0$ iff $x^{m-1}(1-x) \leq \frac{1}{k+m}$. It is easy to see that, for $x \in [0, 1-1/e]$, $x^{m-1}(1-x)$ is maximized at $x = 1-1/e$ for $m > 2$. Let $c = \frac{e}{e-1}$, and $m = log_c(2k) + 2$. For this choice of m,

$$x^{m-1}(1-x) - \frac{1}{k+m} \leq \frac{1}{c^{log_c(2k)+1}}\frac{1}{e} - \frac{1}{k + log_c(2k) + 2}$$

which can be seen to be non-positive for all k. This proves the following lemma.

Lemma 7. *Given any D and k, let $m = log_{\frac{e}{e-1}}(2k) + 2 \simeq \frac{\log k}{\log \frac{e}{e-1}} + 3.5$. Then,*

$$\text{Gain}_D - \text{Loss}_D \geq 0$$

Thus, we have established the following theorem.

Theorem 8 (One-item Upper Bound). *For the case of selling one item, we have*

$$\text{Eff}(\text{Rma}(k+m)) \geq \text{Eff}(\text{Ema}(k))$$

for any k and $m \geq \left\lfloor \log_{\frac{1}{\alpha}} 2k \right\rfloor + 2$, where $\alpha = 1 - 1/e$. Thus, $O(\log k)$ extra bidders suffice for the revenue maximizing mechanism to achieve at least as much efficiency as the efficiency maximizing mechanism with k bidders.

3.3 Lower bound on the number of extra bidders required

In this section, we will prove a lower bound on the number of extra bidders m needed for the revenue maximizing mechanism to achieve at least as much efficiency as the efficiency maximizing mechanism with k bidders. To prove such a lower bound, it suffices to specify a distribution D s.t. the contribution of the m extra bidders to expected efficiency of Rma$(k+m)$ is no more than Gain and show that Gain − Loss < 0 for this choice of k and m.

Consider the distribution $G_{\phi,r}$ for any choice of r, ϕ (see definition 2), and arbitrarily small ϵ. We first show that for the distribution specified by $G_{\phi,r}(x)$, the contribution of m extra bidders to the efficiency of Rma$(k+m)$ is arbitrarily close to $\text{Gain}_{G_{\phi,r}}$. To see this, note that the maximum value possible under distribution $G_{\phi,r}(x)$ is $r+\epsilon$; thus, the maximum possible efficiency for any draw of bidder values is $r+\epsilon$. When all the first k bidders draw a value below r, the m extra bidders contribute a maximum of $r+\epsilon$ to the efficiency of Rma$(k+m)$ with probability equal to $(1-G^m_{\phi,r})$. Thus, the total contribution of the m extra bidders to the efficiency of Rma$(k+m)$ is arbitrarily close to $r(1-G^m_{\phi,r})$, which is the same as Gain.

Let $\alpha = 1-1/e$. Let $m(k) = \left\lfloor \log_{1/\alpha}(k+1)(1-\alpha) \right\rfloor + 1$. Next we'll show that $\text{Gain}_{G_{\phi,r}} - \text{Loss}_{G_{\phi,r}} < 0$ for some choice of ϕ, any k and for all $m \leq m(k)$. To do this, recall the polynomial $q(x) = x^{k+m} + \ln(1-x) + \sum_{i=1}^{k} \frac{x^i}{i}$. By Equation 3, we know that $\text{GAIN}_{G_{\phi,r}} - \text{LOSS}_{G_{\phi,r}} = \frac{-rq(\phi)}{\phi^k}$. Therefore, we just need to show that $q(x) > 0$ for the above choice of m and some x (we will choose ϕ to be that x).

$$
\begin{aligned}
q(\alpha) &= \alpha^{k+m(k)} + \ln(1-\alpha) + \sum_{i=1}^{k} \frac{\alpha^i}{i} \\
&= \alpha^{k+m(k)} - \sum_{i=1}^{\infty} \frac{\alpha^i}{i} + \sum_{i=1}^{k} \frac{\alpha^i}{i} \\
&= \alpha^{k+m(k)} - \sum_{i=k+1}^{\infty} \frac{\alpha^i}{i} \\
&> \alpha^{k+m(k)} - \frac{1}{k+1} \sum_{i=k+1}^{\infty} \alpha^i \\
&= \alpha^{k+m(k)} - \frac{\alpha^{k+1}}{(k+1)(1-\alpha)} \\
&\geq 0
\end{aligned}
$$

where the last inequality follows from the choice of m.

This proves the following theorem.

THEOREM 9 (ONE-ITEM LOWER BOUND). *Let $\alpha = 1 - 1/e$ and let $m(k) = \left\lfloor \log_{1/\alpha}(k+1)(1-\alpha) \right\rfloor + 1$. Then, if bidders are drawn i.i.d. from the distribution described by $G_{\alpha,r}$, $\text{EFF}(\text{RMA}(k+m)) < \text{EFF}(\text{EMA}(k))$ for any r, any k and any $m \leq m(k)$. In other words, $m(k)$ extra bidders do not suffice for the revenue maximizing mechanism to achieve as much efficiency as the efficiency maximizing mechanism with k bidders.*

Note that $m(k) \simeq \frac{\log k+1}{\log \frac{e}{e-1}} - 2.2$, which differs from the upper bound only by a small additive constant, namely 5.7.

4. EXTENSION TO MULTIPLE ITEMS

In this section we consider the case of selling t identical items. As before, the efficiency maximizing mechanism gets k bidders, and we want to find the smallest number of extra bidders that suffice for the revenue maximizing auction to make as much efficiency. As seen in the previous section, $m = m(k) = \left\lfloor \log_{\frac{1}{\alpha}} 2k \right\rfloor + 2$ suffice when selling only one item. So clearly, mt extra bidders would suffice in the case of t identical items. However, as we show below, it is possible to prove a much tighter bound.

The question is to find the smallest number $s = s(k)$ such that the efficiency of the revenue maximizing mechanism with $k+m+s$ bidders, $\text{RMA}(k+m+s)$, is at least as much as the efficiency of the efficiency maximizing mechanism on k bidders, $\text{EMA}(k)$, where $m = m(k)$. As before, to compare the two mechanisms with different number of bidders, we will think of the process of drawing the values of $k+m+s$ bidders $(b_1, b_2, ..b_{k+m+s})$ independently from the given distribution D; the first k bidders $(b_1, b_2, ..b_k)$ participate in both $\text{RMA}(k+m+s)$ and $\text{EMA}(k)$, whereas the last $m+s$ bidders participate only in $\text{RMA}(k+m+s)$.

We partition the space of draws of bidder values into $k+1$ parts: For $i = 0, ..., k$, the ith part, Ω_i, consists of those draws in which exactly i of the first k bidders have values greater than r_D. We now focus on one of these parts, say, the ith part Ω_i, and try to determine the expected efficiency of $\text{EMA}(k)$ and $\text{RMA}(k+m+s)$ over this restricted space. Wlog, we may assume that $i < t$ (if $i \geq t$, then the efficiency of $\text{RMA}(k)$ is already equal to that of $\text{EMA}(k)$). We define $t' = t - i$ and $k' = k - i$.

Let $(b_{max_1}, b_{max_2}, \cdots, b_{max_k})$ be the bids of first k bidders in the decreasing order of their value. Also let Γ_i denote the sum of the highest i of these bids, conditioned on being in Ω_i.

The expected efficiency of $\text{EMA}(k)$ conditioned on being in Ω_i, $\text{EFF}(\text{EMA}^i(k))$, equals

$$
\begin{aligned}
&= E[\sum_{j=1}^{i} b_{max_j} \mid \Omega_i] + E[\sum_{j=i+1}^{k} b_{max_j} \mid \Omega_i] \\
&\leq \Gamma_i + t' E[b_{max_{i+1}} \mid \Omega_i]
\end{aligned}
$$

Now,

$$
\begin{aligned}
&E[b_{max_{i+1}} \mid \Omega_i] = \\
&E[max(b_{i+1}, b_{i+2}, \cdots, b_k) \mid b_1..b_i \geq r \ \& \ b_{i+1}..b_k < r] \\
&\qquad \text{(because of the symmetry)} \\
&= E[max(b_{i+1}, b_{i+2}, \cdots, b_k) \mid \ b_{i+1}..b_k < r] \\
&\qquad \text{(since } b_j's \text{ are independent)} \\
&= \frac{\int_0^r x f^{(k')}(x) dx}{F^{k'}(r)}
\end{aligned}
$$

Therefore,

$$
\text{EFF}(\text{EMA}^i(k)) \leq \Gamma_i + t' \frac{\int_0^r x f^{(k')}(x) dx}{F^{k'}(r)} \tag{4}
$$

Conditioned on being in Ω_i, the contribution to the expected efficiency of $\text{RMA}(k+m+s)$ from the first k bidders is Γ_i. The contribution to the expected efficiency of $\text{RMA}(k+m+s)$ from the remaining $m+s$ bidders depends on how many of the extra bidders have value above r. If j of these bidders have value more than r, then the contribution is at least $\min(j, t')r$. We have the following lower bound on this contribution (note that the contribution from the extra $m+s$ bidders is independent of the first k bidders, and hence of Ω_i):

LEMMA 10. *For any $\epsilon > 0$, and for large enough m, if $s \geq (t + (1+\epsilon)t \log m)$, then the total contribution to the efficiency of $\text{RMA}(k+m+s)$ from the remaining $m+s$ bidders is at least $rt'(1 - F^m(r))$.*

PROOF. Recall that $\phi = F(r)$. Let $a_j = \binom{m+s}{j} \phi^{m+s-j} (1-\phi)^j$. The contribution is:

$$
\begin{aligned}
&r \left[\sum_{j=0}^{t'-1} j a_j + (1 - \sum_{j=1}^{t'-1} a_j) t' \right] \\
= \ &r \left[t' - \sum_{j=0}^{t'-1} a_j (t' - j) \right]
\end{aligned}
$$

But,

$$
\begin{aligned}
a_j(t'-j) &\leq \frac{(m+s)^j}{j!} \phi^m \phi^{s-j} (1-\phi)^j (t'-j) \\
&\leq \phi^m (m+s)^j (1-1/e)^s (t'-j) \\
&\leq \phi^m
\end{aligned}
$$

for $s \geq (t + (1+\epsilon)t \log m) + \log t$ and any $\epsilon > 0$ and large enough m. Thus the contribution is at least $rt'(1-\phi^m)$. $\square$

Let $m = log_{e/(e-1)}(2k) + 2$. Then, using Lemma 10 and the discussion above, the efficiency of RMA conditioned on being in Ω_i is:

$$\begin{aligned}
\text{EFF}(\text{RMA}^i(k+m+s)) &\geq \Gamma_i + t'(1-F(r)^m)r \\
&\geq \Gamma_i + t' \frac{\int_0^r x f^{(k')}(x)dx}{F^{k'}(r)} \\
&\text{(by Lemma 7, and since } k' \leq k) \\
&\geq \text{EFF}(\text{EMA}^i(k)) \\
&\quad \text{(by equation (4))}
\end{aligned}$$

Thus we have proved the following theorem (we have not tried to optimize the constants or how large k has to be for this result to hold).

THEOREM 11 (MULTI-ITEM UPPER BOUND). *In the case of selling t identical items, we have*

$$\text{EFF}(\text{RMA}(k+m+s)) \geq \text{EFF}(\text{EMA}(k))$$

for $m \geq \left\lfloor \log_{\frac{1}{\alpha}} 2k \right\rfloor + 2$ and $s \geq (t + (1+\epsilon)t \log m)$, for every $\epsilon > 0$ and large enough k. Thus, approximately, $\log k + t \log\log k$ extra bidders suffice.

5. THE CASE OF REGULAR DISTRIBUTION

In this section, we will show that for any given k and m, there exists a regular distribution D s.t. expected efficiency of RMA(k+m) is less than the expected efficiency of EMA(k).

To recall, a distribution D is regular if and only if the function $\psi(x) := x - \frac{1}{h_D(x)}$ is non decreasing in x. Now consider the following distribution:

$$P_{\epsilon,r}(x) := \begin{cases} 1 - \frac{\epsilon}{x+\epsilon} & 0 \leq x < r, \\ 1 & x \geq r \end{cases}$$

One can easily verify that the above distribution is regular for every choice of $\epsilon, r > 0$, by evaluating its ψ function. Moreover the reserve price of this distribution equals r. Now, using similar arguments as used in the previous sections, we can show that the contribution of the extra m bidders to EFF[RMA($k+m$)] is $r(1-(1-\frac{\epsilon}{r+\epsilon})^m)(1-\frac{\epsilon}{r+\epsilon})^k \leq r(1-(1-\frac{\epsilon}{r+\epsilon})^m)$.

Also, the extra contribution of the first k bidders to EFF[EMA(k)] over EFF[RMA($k+m$)] when all of the first k bidders have a value below reserve price r is $\int_0^r x[P^k_{\epsilon,r}(x)]'dx$.

Now, as we decrease the value ϵ, the term $r(1-(1-\frac{\epsilon}{r+\epsilon})^m)$ decreases and $\int_0^r x[P^k_{\epsilon,r}(x)]'dx$ increases for a fixed k and m. Moreover, one can show that there exists a small enough $\epsilon := \epsilon'$ such that $\int_0^r x[P^k_{\epsilon',r}(x)]'dx$ is more than $r(1-(1-\frac{\epsilon'}{r+\epsilon'})^m)$. Thus, for the distribution $P_{\epsilon',r}$, the loss in EFF[RMA(k+m)] because of the reserve price is more than the gain from extra m bidders.

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