Quiz 1, Mon 09-26-11
CS 2050, Intro Discrete Math for Computer Science

This quiz has 10 pages (including this cover page) and 5 Problems:
Problems 1, 2, 3 and 4 are mandatory (2 pages each.)
Problem 5 is optional, for extra credit.
You have 50 minutes.

Fill out your information on this cover page FIRST.

Last Name ...............................................................

First Name ...............................................................  

email .................................................................
Problem 1: 25 Points
Recall the definition of divisibility (here stated, for simplicity, only for positive integers): For positive integers \(a\) and \(d\), we say that \(a\) is divisible by \(d\) and use the notation \(d|a\) if and only if there exists a positive integer \(c\) such that \(a = d \times c\). Equivalently:

\[
\forall a \in \mathbb{N}, \forall d \in \mathbb{N} \quad (d|a \iff \exists c \in \mathbb{N}: a = d \times c)
\]

For (a) and (b) below: True or false?
If true give a proof, if false give a counter-example:

(a) \(\forall a \in \mathbb{N}, \forall b \in \mathbb{N}, \forall d \in \mathbb{N} \quad (d|a \text{ and } d|b) \implies d|(a \times b)\) .

(b) \(\forall a \in \mathbb{N}, \forall b \in \mathbb{N}, \forall d \in \mathbb{N} \quad (d|(a \times b) \implies (d|a \text{ or } d|b)\) .

Problem 1: Answer Part (a) - go to the next page for Part (b).

True, Proof:
From the definition of divisibility we have:

\[
d|a \implies \exists c_1 \in \mathbb{N}: a = d \times c_1 \quad (1)
\]

\[
d|b \implies \exists c_2 \in \mathbb{N}: b = d \times c_2 \quad (2)
\]

Combining the right-hand-sides of (1) and (2) by multiplying \(a\) and \(b\) we have:

\[
a \times b = (d \times c_1) \times (d \times c_2)
\]

\[
= d \times (c_1 \times d \times c_2)
\]

\[
= d \times c \quad \text{for } c = c_1 \times d \times c_2
\]

Therefore, from the definition of divisibility, we infer that \(d|(a \times b)\).
Problem 1: Answer Part (b)

False, Counter-Example:

\[ a = 10 \quad \text{Note: I thought of } 10 \text{ as } 10 = 2 \times 5. \]
\[ b = 15 \quad \text{Note: I thought of } 15 \text{ as } 15 = 3 \times 5. \]

Note: My strategy was for \( a \) and \( b \) to have two factors that are relatively prime, such as 2 and 3. Then the product of these factors (in this case \( 2 \times 3 = 6 \)) would obviously divide \( a \times b \) (in this case \( 10 \times 15 = 2 \times 5 \times 3 \times 5 = 2 \times 3 \times 5 \times 5 = 6 \times 25 \)), but it would not divide by itself, neither \( a \) nor \( b \).

\[ a \times b = 150 \]
\[ d = 6 \]

Obviously, 6 divides \( a \times b = 150 \), since \( 150 = 6 \times 25 \).

But 6 does not divide \( a = 10 \) and 6 does not divide \( b = 15 \).
Problem 2: 25 Points
(a) Argue that, for every positive integer $n$, and for all $x$ such that $0 < x < 1$,

$$\sum_{k=0}^{n} x^k < \frac{1}{1-x}.$$  

(b) Argue that, for every positive integer $n$,

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k < 1.$$  

Problem 2: Answer Part (a) - go to the next page for Part (b).

We know that, for all $x \neq 0$, $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$.

We may therefore write:

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} < \frac{1}{1-x}$$

where the last inequality follows since, for $0 < x < 1$, we have $x^{n+1} > 0$, $(1-x) > 0$,

therefore the fraction $\frac{x^{n+1}}{1-x}$ is a positive number.
Problem 2: Answer Part (b)

\[
\sum_{k=1}^{n} \left( \frac{1}{2} \right)^k = -\left( \frac{1}{2} \right)^0 + \left( \frac{1}{2} \right)^0 + \sum_{k=1}^{n} \left( \frac{1}{2} \right)^k \\
= -1 + \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k \\
< -1 + \left( \frac{1}{1 - \frac{1}{2}} \right) \quad \text{using the bound of part (a) for } x = \frac{1}{2} \\
= -1 + 2 \\
= 1 .
\]
Problem 3: 25 Points

(a) Argue that, for every positive integer $n$,

$$
\sum_{k=1}^{n} \left( k - \frac{1}{2} \right) = \frac{n^2}{2}.
$$

(b) Evaluate in terms of $n$ the following three products:

(assume that $n$ is a power of 2, thus $\log_2 n$ is integer and show, in one line, your work)

(i) $\prod_{i=1}^{\log_2 n} 2 = ?$

(ii) $\prod_{i=1}^{\log_2 n} 4 = ?$

(iii) $\left( \prod_{i=1}^{\log_2 n} 2 \right)^2 = ?$

Problem 3: Answer Part (a) - go to the next page for Part (b).

We know that, for every positive integer $n$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

We may now write:

$$
\sum_{k=1}^{n} \left( k - \frac{1}{2} \right) = \left( \sum_{k=1}^{n} k \right) - \left( \sum_{k=1}^{n} \frac{1}{2} \right)
$$

$$
= \frac{n(n+1)}{2} - \frac{n}{2}
$$

$$
= \frac{n^2 + n - n}{2}
$$

$$
= \frac{n^2}{2}
$$
Problem 3: Answer Part (b): (i), (ii) and (iii)

(i) \( \prod_{i=1}^{\log_2 n} 2 = 2^{\sum_{i=1}^{\log_2 n} 1} = 2^{\log_2 n} = n \).

(ii) \( \prod_{i=1}^{\log_2 n} 4 = 4^{\sum_{i=1}^{\log_2 n} 1} = 4^{\log_2 n} = 2^{2\log_2 n} = 2^{\log_2 n \times 2} = n^2 \).

(iii) \( \left( \prod_{i=1}^{\log_2 n} 2 \right)^2 = (n)^2 = n^2 \), where the middle equality follows from (i).
Problem 4: 25 Points
(a) Show that gcd(14039,1529)=139. Use Euclid’s algorithm and show your work.
(b) Argue that a $1529 \times 14039$ board can be fully covered using $139 \times 139$ tiles.

Problem 4: Answer Part (a) - go to the next page for Part (b).

\[
\begin{align*}
14039 & = 9 \times 1529 + 278 \\
1529 & = 5 \times 278 + 139 \\
278 & = 2 \times 139 + 0
\end{align*}
\]
Problem 4: Answer Part (b)

The key observation is that 139 divides both 1529 and 14039. In particular, 1529 = 11 \times 139 and 14039 = 101 \times 139. We may therefore cover the 1529 \times 14039 board with 139 \times 139 tiles as follows: We place 101 tiles horizontally starting from the top left, and repeat this (placing 101 tiles horizontally) 11 times vertically. We will need, in total 101 \times 11 = 1111 tiles (of dimension 139 \times 139).
Problem 5: Optional, Extra Credit

(a) Let \( m > 1 \) and \( n \geq 1 \) be arbitrary integers.

\[
\sum_{k=1}^{n} \left( \frac{1}{m} \right)^k < ?
\]

Problem 5: Answer Part (a)

\[
\begin{align*}
\sum_{k=1}^{n} \left( \frac{1}{m} \right)^k &= - \left( \frac{1}{m} \right)^0 + \left( \frac{1}{m} \right)^0 + \sum_{k=1}^{n} \left( \frac{1}{m} \right)^k \\
&= -1 + \sum_{k=0}^{n} \left( \frac{1}{m} \right)^k \\
&< -1 + \left( \frac{1}{1 - \frac{1}{m}} \right) \text{ using the bound of part (a) for } x = \frac{1}{m} \\
&= -1 + \frac{m}{m-1} \\
&= \frac{-m + 1 + m}{m-1} \\
&= \frac{1}{m-1}
\end{align*}
\]

(b) Let \( m > 1 \) and \( n > 1 \) be arbitrary integers.

How does the tiling statement of Problem 4 generalize?

Problem 5: Answer Part (b)

For arbitrary positive integers \( m \) and \( n \), if \( d \) is a positive integers that divides both \( m \) and \( n \) (ie \( m = d \times c_1 \) and \( n = d \times c_2 \)), then the \( m \times n \) board can be covered using \( d \times d \) tiles.