Problem 1

(a) Prove that if at least one of $n$ or $m$ is even then the product $n \times m$ is even.

(b) Prove that if both $n$ and $m$ are odd then the product $n \times m$ is odd.

Answer

(a) Direct proof, based on the general fact that:

An integer is even, if and only if it is of the form $2k$, for some integer $k$.

For this question, we may assume, without loss of generality, that $n$ is even. Therefore, by the above general fact, $n = 2k$, for some integer $k$. Therefore, the product $n \times m = (2k) \times m = 2(k \times m) = 2k'$, where $k'$ is the integer $k' = k \times m$. Therefore, by the same above general fact, $n \times m$ is even.

(b) Direct proof, based on the general fact that:

An integer is odd, if and only if it is of the form $2k+1$, for some integer $k$.

By the above general fact, $n = 2k+1$, for some integer $k$. Also, by the above general fact, $n = 2l+1$, for some integer $l$. Therefore, the product $n \times m$ is

\[
\begin{align*}
  n \times m &= (2k + 1) \times (2l + 1) \\
  &= 4kl + 2k + 2l + 1 \quad \text{by simple calculations} \\
  &= 2 \times (2kl + k + l) + 1 \quad \text{factoring out the 2 from the first 3 terms} \\
  &= 2k' + 1 \quad \text{where $k'$ is the integer $k' = 2kl + k + l$}
\end{align*}
\]

Therefore, by the same above general fact that an integer is odd, if and only if it is of the form $2k+1$, for some integer $k$, we conclude that $n \times m = 2k' + 1$ is odd.
**Problem 2**

(a) Let $S_n = \sum_{i=1}^{n}i$. Prove that, for every $n \geq 2$, $S_n > n$.

(b) Prove that there is unique positive integer that equals the sum of the positive integers not exceeding it.

**Answer**

(a) Direct proof, based on the fact that $S_n = \sum_{i=1}^{n}i = \frac{n(n+1)}{2}$, $\forall i \geq 1$.

In the sequence of “if and only if” ($\iff$) inferences below, we assume that $n > 0$ is an integer, and proceed to compare $\frac{n(n+1)}{2}$ with $n$.

\[
\begin{align*}
\frac{n(n+1)}{2} &> n \\
n(n+1) &> 2n \\
n + 1 &> 2, \text{ where we cancelled } n \text{ from both sides, since } n > 0 \\
n + 1 &> 1 + 1 \\
n &> 1 \\
n &\geq 2, \text{ since } n \text{ is an integer.}
\end{align*}
\]

We therefore conclude that

\[ (n \geq 2) \implies \left( \frac{n(n+1)}{2} > n \right) \]

and since

\[ S_n = \frac{n(n+1)}{2}, \forall i \geq 1 \quad \text{(consequently also } \forall i \geq 2) \]

we infer

\[ (n \geq 2) \implies (S_n > n). \]

Which is equivalent to

\[ S_n > n, \quad \forall n \geq 2. \]

(b) From part (a) we have

\[ S_n = \sum_{i=1}^{n}i > n, \quad \forall n \geq 2, \]

meaning that, every positive integer strictly greater than 1 is strictly smaller than the sum of the positive integers not exceeding it. Therefore, the number 1 is the only possible integer which might be equal to the sum of the positive integers not exceeding it.

Indeed, we may verify that

\[ S_1 = \sum_{i=1}^{1}i = 1. \]
Problem 3
Let \( S_n \) be the sum of all positive integers from 1 to \( n \), ie \( S_n = 1 + 2 + \ldots + n \) or \( S_n = \sum_{i=1}^{n} i \).
Let \( S'_n \) be the sum of the squares of all positive integers from 1 to \( n \), ie \( S'_n = 1^2 + 2^2 + \ldots + n^2 \) or \( S'_n = \sum_{i=1}^{n} i^2 \).
Let
\[
S''_n = 1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1)
\]
Prove that \( S''_n = S_n + S'_n \).

Answer
\[
S''_n = \sum_{i=1}^{n} i(i+1)
\]
\[
= \sum_{i=1}^{n} (i(i+1))
\]
\[
= \sum_{i=1}^{n} (i^2 + i) , \text{ by calculations inside the sum}
\]
\[
= \left( \sum_{i=1}^{n} i^2 \right) + \left( \sum_{i=1}^{n} i \right) , \text{ breaking up the terms of the sum}
\]
\[
= S'_n + S_n , \text{ by definition of } S'_n \text{ and } S_n .
\]
Problem 4
Prove that, for every positive integer \( n \), \( \sum_{i=1}^{2n} (1 + (-1)^i) = 2n \).

Answer
Direct argument, based on two basic facts:

First Fact
\[
\begin{align*}
( \text{if } i \text{ is even then } (-1)^i &= 1 ) & \Rightarrow & (1 + (-1)^i) &= 2 \\
( \text{if } i \text{ is odd then } (-1)^i &= -1 ) & \Rightarrow & (1 + (-1)^i) &= 0
\end{align*}
\]

Second Fact
There are \( n \) even numbers between 1 and \( 2n \).

Combining Facts 1 and 2 above, we may write:
\[
\sum_{i=1}^{2n} (1 + (-1)^i) = n \times 2 = 2n .
\]
Problem 5
(a) Prove that the $5 \times 5$ board with the top left corner removed can be covered using $2 \times 1$ tiles.
(b) Prove that, for any odd integers $n > 1$ and $m > 1$, the $n \times m$ board with the top left corner removed can be covered using $2 \times 1$ tiles.

Answer TO BE ADDED
Problem 6: Extra Credit

Let $n > 1$ and $m > 1$ be odd integers. Let $S$ be the set of squares of the $n \times m$ board. Give a complete characterization of the (single) squares that, if removed from the $n \times m$ board, then the remaining $(n \times m) - 1$ area can be covered using $2 \times 1$ tiles. That is, characterize the set $T \subseteq S$ such that

\[
\begin{align*}
    x \in T & \implies S \setminus \{x\} \text{ can be covered using } 2 \times 1 \text{ tiles} \\
    x \in S \setminus T & \implies S \setminus \{x\} \text{ cannot be covered using } 2 \times 1 \text{ tiles}
\end{align*}
\]

Answer TO BE ADDED