Let us prove Theorem 1:

**Theorem 1** For any $\delta > 0$ \[ \Pr \left[ X > (1+\delta)\mu \right] < \left( \frac{e^\delta}{(1+\delta)\mu} \right)^\mu \]

**Proof** Let $t$ be a real number (to be determined later, as convenient so as to optimize the bound):

\[ \Pr \left[ X > (1+\delta)\mu \right] = \Pr \left[ e^{tX} > e^{t(1+\delta)\mu} \right] \]

By Markov's Inequality

\[ \leq \frac{E \left[ e^{tX} \right]}{e^{t(1+\delta)\mu}} \]

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**Note:**

Realize that $e^{tX}$ is a random variable and we need to compute $E \left[ e^{tX} \right]$ to proceed with the bound.
Computing $E[e^{tx}]$:

$$E[e^{tx}] = E[e^{tx_1 + ... + tx_n}] = E[e^{tx_1}]E[e^{tx_2}]...E[e^{tx_n}]$$

So let us compute $E[e^{tx_i}]$:

$$E[e^{tx_i}] = e^{tp_i} + e^{t(1-p_i)}$$

$$= e^{tp_i} + 1-p_i$$

$$= 1 + p_i(e^t - 1)$$

$$p_i(e^t - 1) < e$$

Using the fact $1 + x < e^x$ and substituting $p_i(e^t - 1)$ for $x$.

Note:

- Realize that each one of the $e^{tx_i}$'s is also a random variable.

- In general, $E[Y_1Y_2] \neq E[Y_1]E[Y_2]$. This is because the $Y_i$'s might be dependent.

- However, Lemma: If $Y_1$ and $Y_2$ are independent then $E[Y_1Y_2] = E[Y_1]E[Y_2]$.

- Now because the $e^{tx_i}$'s are independent, we can write that the expectation of their product is the product of their expectations.
Combining (1), (2) and (3) we have:

\[ P_r \left[ X > (1 + \delta) \mu \right] < \frac{E \left[ e^{tX_i} \right]}{e^{t(1 + \delta) \mu}} \]

By (1)

\[ = \prod_{i=1}^{n} E \left[ e^{tX_i} \right] \]

By (2)

\[ < \prod_{i=1}^{n} P_i (e^t - 1) \]

By (3)

\[ < \prod_{i=1}^{n} e^{(e^t - 1) \sum P_i} \]

By calculation

\[ = e^{(e^t - 1) \mu} \]

By the definition

\[ \mu = \sum_{i=1}^{n} P_i \]
\[ \begin{align*}
\text{P} & = \ln(1+\delta) \\
& = \frac{e^{(\ln(1+\delta))} - 1}{\ln(1+\delta)} \mu \\
& = e \frac{1+\delta - 1}{(1+\delta)(1+\delta)} \mu \\
& = e \left( \frac{e}{(1+\delta)^{1+\delta}} \right)^{\mu} \\
\end{align*} \]

By setting \( t = \ln(1+\delta) \)

QED.