Balancing Balls and Bins

Each ball thrown independently and uniformly at random into one of the \( m \) bins.

**Theorem**

Let \( X_j \) be the number of balls that ended in bin \( j \).

(a) If \( n = m \) then 
\[
\Pr \left[ \exists j : X_j > \frac{2 \ln n}{\ln \ln n} \right] < \frac{1}{\eta} = \frac{1}{m}
\]

(b) If \( n = m \ln m \) then 
\[
\Pr \left[ \exists j : X_j > 3.75 \ln m \right] < \frac{1}{m}
\]

(c) If \( n = m^2 \) then 
\[
\Pr \left[ \exists j : X_j > \left(1 + 2 \sqrt{\frac{2 \ln m}{m}}\right) m \right] = \Pr \left[ \exists j : X_j > m + 2 \sqrt{m \ln m} \right] < \frac{1}{m}
\]
**Proof**

Let \( p_{ij} \) be the probability that ball \( i \) ends up in bin \( j \).

\[
\begin{align*}
p_{ij} &= \frac{1}{m}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m \\
E[X_j] &= \sum_{i=1}^{n} p_{ij} = \frac{n}{m} = \mu, \quad 1 \leq j \leq m
\end{align*}
\]

By Chernoff Bounds

\[
\Pr \left[ X_j > (1+\delta) \mu \right] < \left( \frac{e}{(1+\delta)^{1+\delta}} \right)^{\mu}
\]

(a) For \( n=m \) (hence \( \mu=1 \)) and \( \delta = \frac{2\ln n}{\ln \ln n} - 1 \) \( \star \) implies:

\[
\Pr \left[ X_j > \frac{2\ln n}{\ln \ln n} \right] < \frac{1}{e} \left( \frac{1}{e^{\frac{2\ln n}{\ln \ln n}}} \right)^{\frac{2\ln n}{\ln \ln n}} = \frac{1}{e} \left( \frac{1}{e} \right)^{\frac{2\ln n}{\ln \ln n}} = \frac{1}{e^{\frac{1}{\ln n}}} < \frac{1}{n^2} < \frac{1}{m^2}
\]
(b) For \( n = m^2 \) (hence \( \mu = \ln m \)) and \( \delta = 2.75 \) \( \star \) implies:

\[
\Pr[X_j > (1+\delta) \mu] = \Pr[X_j > 3.75 \ln m] < \left( \frac{e^{\frac{8}{(1+\delta)(1+\delta)}} \ln m}{\frac{e}{3.75}} \right)^{\ln m} < \left( \frac{1}{e^2} \right)^{\ln m} = \frac{1}{m^2}
\]

(c) For \( n = m^2 \) (hence \( \mu = m \)) and \( \delta = 2 \sqrt{2 \ln m / m} \ll 1 \)

we will use the bound

\[
\Pr[X_j > (1+\delta) \mu] < e^{-\frac{\delta^2}{4} m}
\]

which gives

\[
\Pr[X_j > (1 + 2 \sqrt{2 \ln m / m}) m] < e^{-4 \cdot 2 \ln m / m} \cdot \frac{1}{4} m = e^{-2 \ln m} = \frac{1}{m^2}
\]

Hence, for all cases (a), (b), and (c) we have bounded the probability that any particular \( X_j \) exceeds the stated value by \( \frac{1}{m^2} \). Now given that there are \( m \) bins, we can use union bound and bound the probability that there exists a bin with more balls than in the stated value by \( \frac{1}{m} \)

Q.E.D.