Simultaneous Recovery of Shape, Motion and Grouping by Applying Rank Constraints

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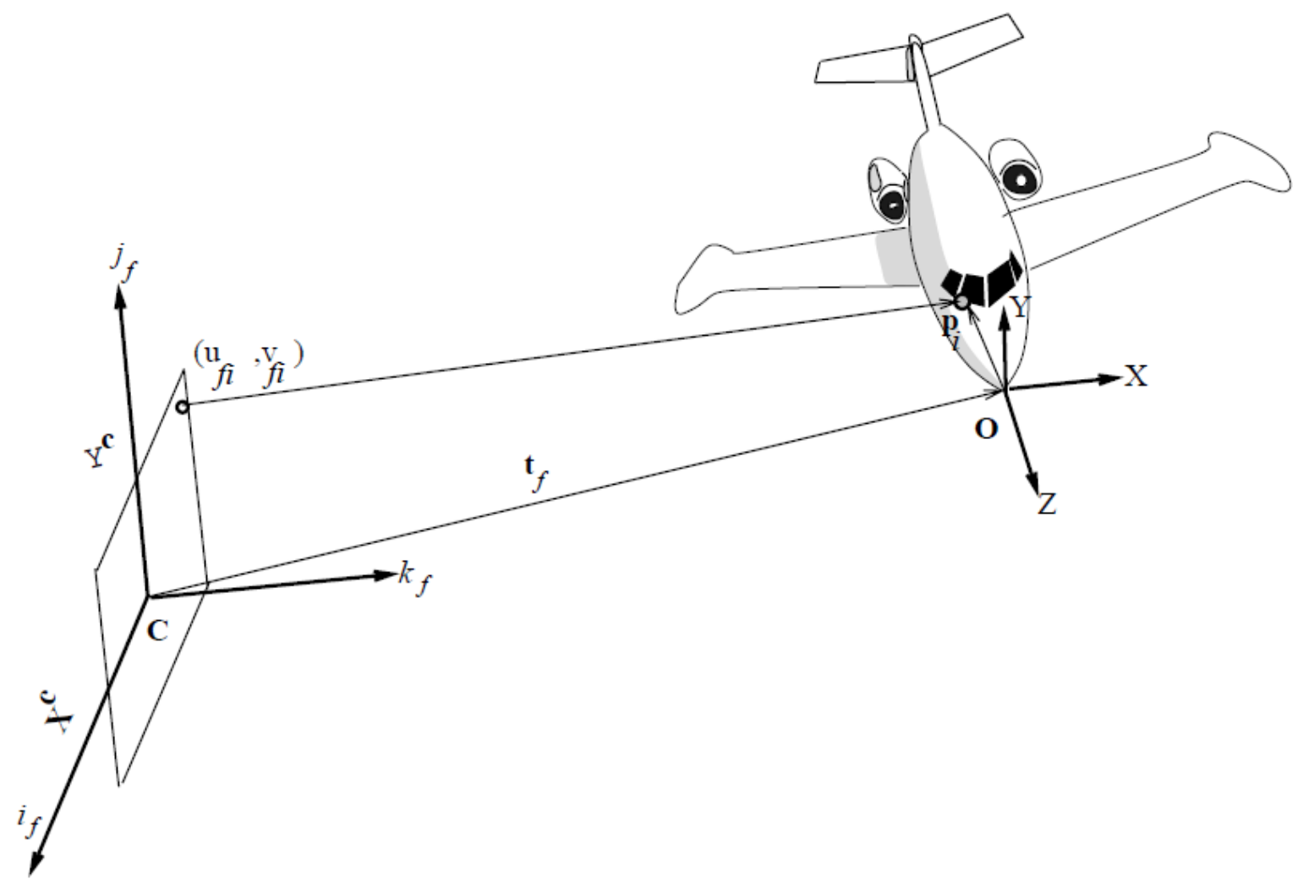
Motivation



- *Two Object Moving Rigidly and Independently in $\mathbb{R}^3$.*
- *Each object has more than 5 points which do not lie on a plane.*
- *An Image Plane C perpendicular to Z axis and passing from origin.*
- *The objects are arbitararily but independently rotated and translated F times in space.*
- *We are given the projection of object points to C at every instance.*
- *Goal 1: Group the points into two subsets corresponding to each object.*
- *Goal 2: recover location of object points in their local coordinate frame.*
- *Goal 3: recover the $F \times 2$ rotation and translations in X and Y axis.*

Previous Work

The most related works:

- *Tomasi and Kanade (IJCV 92), use factorization and rank constraints to recover shape and motion for a single object.*
- *Costeira and Kanade (IJCV 98), extend Tomasi's work to recover the point groupings for multiple objects.*
- *Kanatani (ICCV 01), proves Costeira's method works in theory.*

Problem Formulation

- *Two set of points $\mathcal{P}_1 = \{P_1, ..., P_{n_1}\}$ and $\mathcal{P}_2 = \{Q_1, ..., Q_{n_2}\}$ in $\mathbb{R}^3$.*
- *Each set has a local coordinate system with its origin at the mean of its points.*
- *Each point set is rigid meaning its local location of points $\hat{P}_i$ are always constant.*
- *Point sets are independently transformed F times. Each transformation is a 4×4 matrix which consists of a rotation and a translation and has rank 4.*
- *The input is projection of all $N = N_1 + N_2$ points to C at every instance $f \in \{1, .., F\}$.*
- *The projections are sorted, i.e. the ith projection in all instances corresponds to the same point.*
- *We show it is possible to group the points based on their projections into two set $\mathcal{P}_1$ and $\mathcal{P}_2$.*
- *It is possible to recover the point locations in their local coordinate frames.*
- *It is possible to recover the x and y rows of transformation matrices.*

Approach

The projection of a point P_i at instance f is given by

$$P^C_{f,i} = R_f \hat{P}_i + t_f$$

Homogeneous Coordinates:

$$\begin{aligned} P^C_{f,i} = \begin{bmatrix} u_{f,i} \\ v_{f,i} \end{bmatrix} &= \begin{bmatrix} R_f(2 \times 3) & t_f(2 \times 1) \end{bmatrix} \begin{bmatrix} \hat{P}_i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} I_{x,f} & t_{x,f} \\ J_{x,f} & t_{y,f} \end{bmatrix} \begin{bmatrix} \hat{P}_i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} i_{x,f} & i_{y,f} & i_{z,f} & t_{x,f} \\ j_{x,f} & j_{y,f} & j_{z,f} & t_{y,f} \end{bmatrix} \begin{bmatrix} \hat{p_{x,i}} \\ \hat{p_{y,i}} \\ \hat{p_{z,i}} \\ 1 \end{bmatrix} \end{aligned}$$

Assume we knew the groupings:

$$W_1 = \begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n_1} \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ u_{F,1} & u_{F,2} & \cdots & u_{F,n_1} \\ v_{1,1} & v_{1,2} & \cdots & v_{1,n_1} \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ v_{F,1} & v_{F,2} & \cdots & v_{F,n_1} \end{bmatrix} = \begin{bmatrix} i_{x,1} & i_{y,1} & i_{z,1} & t_{x,1} \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ i_{x,F} & i_{y,F} & i_{z,F} & t_{x,F} \\ j_{x,1} & j_{y,1} & j_{z,1} & t_{y,1} \\ & & \cdot & \\ & & \vdots & \\ j_{x,F} & j_{y,F} & j_{z,F} & t_{z,F} \end{bmatrix} \begin{bmatrix} \hat{p}_{x,1} & & \hat{p}_{x,n_1} \\ \hat{p}_{y,1} & \cdots & \hat{p}_{y,n_1} \\ \hat{p}_{z,1} & & \hat{p}_{z,n_1} \\ 1 & & 1 \end{bmatrix} = M_1 S_1$$

Recovering M and S for one set of points:

We use SVD decomposition to decompose W_1 as $W_1 = U_1 \Sigma_1 V_1^T$. Let $\hat{M}_1 = U\Sigma^{\frac{1}{2}}$ and $\hat{S}_1 = \Sigma^{\frac{1}{2}} V^T$. However $W_1 = (\hat{M}_1 A_1)(A_1^{-1} \hat{S}_1)$ is also a solution.

We will recover A using constraints on rotation and translation.

Given $M_1 = \hat{M}_1 A = \hat{M}_1 [A_R | A_t]$, we apply the following constraints on rotation:

$$\hat{m}_f A_R A_R^T \hat{m}_f^T = 1, s.t. \forall f \in \{1, .., 2F\}$$

$$\hat{m}_f A_R A_R^T \hat{m}_{f+F}^T = 0, s.t. \forall f \in \{1, .., F\}$$

and the following on translation:

$$\bar{W} = \begin{bmatrix} \frac{1}{n_1} \Sigma u_{1,i} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{n_1} \Sigma v_{F,i} \end{bmatrix} = M_1 . \bar{S}_1 = \left[\hat{M}_1 A_R | \hat{M}_1 A_t \right] \begin{bmatrix} \bar{P} \\ 1 \end{bmatrix} = \hat{M}_1 A_t$$

Grouping Points:

$$\begin{aligned} W^* = [W_1 | W_2] &= [M_1 | M_2] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \\ &= [U_1 | U_2] \begin{bmatrix} \Sigma_1^{\frac{1}{2}} & 0 \\ 0 & \Sigma_2^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix} \begin{bmatrix} \Sigma_1^{\frac{1}{2}} & 0 \\ 0 & \Sigma_2^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} V_1^T & 0 \\ 0 & V_2^T \end{bmatrix} \end{aligned}$$

We claim that if we decompose W into $W = U \Sigma V^T$, $N \times N$ matrix $Q = VV^T$ determines the two point sets.

Theorem 1

- *Let $\mathcal{T}$ contain N points and form a matrix $W = [P_1, .., P_N]$ which belongs to r dimensional space $\mathcal{L} \subset \mathbb{R}^n$. Let $Q = V_r V_r^T = \sum_1^r v_i v_i^T$.*
- *Two linearly independent disjoint subsets $\mathcal{T}_1$ and $\mathcal{T}_2$ with rank r_1 and r_2. If P_α belongs to $\mathcal{T}_1$ and P_β belongs to $\mathcal{T}_2$, $Q(\alpha, \beta) = 0$.*

***Proof**:*

- *Suppose $p_1, ..., p_{N_1} \in \mathcal{L}_1$ and $p_{N_1+1}, ..., p_{N_2} \in \mathcal{L}_2$. The null space ($\mathcal{N}_1$) of W_1 will have dimension $v_1 = N_1 - r_1$.*
- *Let $\{n_1, ..., n_{v_1}\}$ be an arbitarary orthonormal basis of $\mathcal{N}_1$ and $\{n'_1, ..., n'_{v_2}\}$ be an arbitarary orthonormal basis of $\mathcal{N}_2$.*
- *$\tilde{n}_i = \begin{pmatrix} n_i \\ 0 \end{pmatrix}$ and $\tilde{n}'_i = \begin{pmatrix} 0 \\ n'_i \end{pmatrix}$. vectors $\{\tilde{n}_1, ..., \tilde{n}_{N_1 - r_1}, \tilde{n}'_1, ..., \tilde{n}'_{N_2 - r_2}\}$ produce an orthonormal system of $\mathbb{R}^N$ belonging to the null space $\mathcal{N}$.*
- *Let $\{v_{r+1}, ..., v_N\}$ be orthonormal right singular vectors of W.*

$$\begin{aligned} [v_{r+1}, ..., v_N] [v_{r+1}, ..., v_N]^T &= \left[\tilde{n}_1, ..., \tilde{n}_{v_1}, \tilde{n}'_1, ..., \tilde{n}'_{v_2}\right] CC^T \left[\tilde{n}_1, ..., \tilde{n}_{v_1}, \tilde{n}'_1, ..., \tilde{n}'_{v_2}\right]^T \\ &= \begin{bmatrix} \tilde{n}_1 & \cdots & \tilde{n}_{v_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \tilde{n}'_1 & \cdots & \tilde{n}'_{v_2} \end{bmatrix} \begin{bmatrix} \tilde{n}_1 & \cdots & \tilde{n}_{v_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \tilde{n}'_1 & \cdots & \tilde{n}'_{v_2} \end{bmatrix}^T = \begin{bmatrix} \Gamma & 0 \\ 0 & \Upsilon \end{bmatrix} \end{aligned}$$

- *If α and β are two rows indices of V, and $\alpha \neq \beta$ we have*

$$\begin{aligned} v_{\alpha 1} v_{\beta 1} + ... + v_{\alpha r} v_{\beta r} & \\ + v_{\alpha(r+1)} v_{\beta(r+1)} \cdots + v_{\alpha N} v_{\beta N} &= 0 \end{aligned}$$

$$v_{\alpha 1} v_{\beta 1} + ... + v_{\alpha r} v_{\beta r} = 0$$

- *As a result the (α, β) rows of matrix $Q = V_r V_r^T$ is zero.*

Theorem 2

Permuting columns of W does not change its set of values.

***Proof:** Assume we swap columns l and m of W and make W'.*

$$\begin{aligned} Q'(\alpha, i) &= V'(\alpha, :) V'(i, :)^T \\ &= V(\beta, :) V(i, :)^T \\ &= Q(\beta, i) \end{aligned}$$

Theorem 3

For every α, there is at least one β were $Q_{\alpha,\beta} \neq 0$.

***Proof:** Assume $W = [W_1 | W_2]$. We will have*

$$Q = V_r V_r^T = \begin{bmatrix} V_{r_1} & 0 \\ 0 & V_{r_2} \end{bmatrix} \begin{bmatrix} V_{r_1} & 0 \\ 0 & V_{r_2} \end{bmatrix}^T = \begin{bmatrix} V_{r_1} V_{r_1}^T & 0 \\ 0 & V_{r_2} V_{r_2}^T \end{bmatrix}$$

We show if there is a $\alpha \in \{1, ..., N_1\}$, there is at least one $\beta \in \{1, ..., N_1\}$ for which

$$v_{\alpha 1} v_{\beta 1} + ... + v_{\alpha r} v_{\beta r} \neq 0$$

If αth row of Q_{r_1} is all zero, but at (α, α), it means row α is independent from all other rows in V_{r_1}.

Means other rows span a subspace with dimension $r_1 - 1$. Means all points $p_1, .., p_{\alpha-1}, p_{\alpha+1}, .., p_{r_1}$ span a subspace with dimension r_1.

This is in contrast with the assumption in Theorem 1 that every r_1 points in $\mathcal{T}_1$ span the r_1 dimensional subspace $\mathcal{L}_1$.